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Semisimplification and Non-Associative Algebras in the Verlinde Category

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Acknowledgements and Statements

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Disclaimer on the use of AI

ChatGPT was used during the writing of this thesis, primarily to improve the structure and clarity of binding text. Its use was limited to rewriting, not writing, certain passages. On occasion, it was also used to ask questions about literature in areas I was less familiar with.

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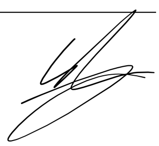

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Contents

0 Introduction

A Dutch introduction and summary can be found in Appendix A.

0.1 Motivation and goals of this thesis

Over the past few decades, the theory of (*symmetric*) *tensor categories* has emerged as a powerful framework providing a formal backbone for a range of structural approaches to commutative algebra and algebraic geometry. Symmetric (or more generally, braided) tensor categories provide a categorical setting in which familiar algebraic objects, such as algebras, modules, and affine group schemes, can be studied in a much more general fashion (see, for example, [Del07; EGNO15; Ven16; Ven23; Ven24; Cou23a; Cou23b]).

A helpful way to understand the conceptual nature of tensor categories is presented in the standard reference [EGNO15], which characterises the subject as "a theory of vector spaces or group representations without vectors," in analogy with how "ordinary category theory may be thought of as a theory of sets without elements."

The study of tensor categories originated with the formulation and investigation of monoidal categories by Saunders Mac Lane in the 1960s (see the classic text [Lan78]), and was subsequently developed further by students of Alexander Grothendieck. A major turning point in the modern theory came with the work of Pierre Deligne [DM82; Del90], whose contributions significantly deepened the understanding of tensor categories.

Since then, tensor categories have become important objects of study, not only as a framework for various constructions in algebra and geometry, but also as subjects of structural and classification theory in their own right (see, for example, [Del02; Ost20; CEO24a; CEOK23; CEO24b]).

In this thesis, we aim to develop the necessary tools to work with *non-associative* algebras in (braided) tensor categories. Recently, in [Kan24], Arun S. Kannan constructed exceptional Lie superalgebras by using Lie algebras in a "non-classical" symmetric tensor category. This realisation is the main motivation behind this thesis, and it shows that non-associative algebras in more exotic categories are worth studying. Aside from constructions related to those given by Kannan and to Lie algebras arising from affine algebraic groups (see [Ven23]), the topic of non-associative algebras in tensor categories remains largely underexplored.

We will mainly be interested in tensor categories over (algebraically closed) fields of positive characteristic, as this is the setting with more exotic behaviour. A perfect example of this exotic behaviour comes from the structure theory of so-called *pre-Tannakian* symmetric tensor categories. Over algebraically closed fields of characteristic zero, Deligne provided a detailed structural classification of these categories in [Del90; Del02]. More specifically, Deligne showed that any such category fibres over the category of super-vector spaces, which implies that it is equivalent to the representation category of an affine group scheme in this category (equivalently, a Hopf algebra in this category). However, the extension of this classification to positive characteristic presented (and still presents) significant obstacles. A key development in the positive characteristic case came from Victor Ostrik, who showed in [Ost20] that any symmetric fusion category over an algebraically closed field of positive characteristic fibres over a particular symmetric tensor category known as the (universal) *Verlinde category* Ver_p, introduced earlier in [GK92; GM94]. This implies that any such category is equivalent to the representation category of an affine group scheme in Ver_p (equivalently, a Hopf algebra).

The Verlinde category arises through the process of *semisimplification* of the category of finite-dimensional representations of the linear algebraic group α_p. Semisimplification, as a categorical operation, relies on the fact that tensor categories possess a unique maximal tensor ideal, thereby resembling a local ring. Taking the quotient of the category by this ideal (which defines a functor) yields a semisimple category called the semisimplification of the original category. It is precisely this semisimplification process that was used by Kannan to construct Lie algebras in Ver_p from classical Lie algebras. As Ver_p contains the category of super-vector spaces, Lie superalgebras can then be obtained by projecting into this category. Similarly, a construction of Kac's 10-dimensional Jordan superalgebra was given in [EEK25] through the use of a Jordan algebra in Ver_p.

We discuss semisimplification in considerable detail in this thesis, going beyond what is currently available in the literature (see [AKO02; EO21a]). In particular, we provide an explicit description of the morphisms in the maximal tensor ideals for categories that are not necessarily pivotal. This is important because it allows us to work with general braided categories that are not necessarily balanced.

Some of our attention will also be devoted to (non-associative) algebras in the Verlinde category, motivated both by its importance in the structure theory and by Kannan's work. We will discuss algebras on simple objects in the Verlinde categories, which turn out to be non-associative algebras (with trivial automorphism groups). A question that captured our attention during the writing of this thesis is the following: "We know that any Lie algebra in Ver_p can be obtained as the semisimplification of an algebra, but can every Lie algebra in Ver_p be obtained as the semisimplification of a Lie algebra?" A negative answer to this question would probably also give a negative answer to [CEO24a, Question 4.6]. Unfortunately, we were not able to answer this question, and it has therefore not taken up much space in this thesis (although it has taken up much of our time).

One remark I would like to make about this thesis is that our primary focus is on developing the necessary background and tools to study non-associative algebras. These tools have not yet been extensively applied within this thesis, but we hope to pursue such applications in the future.

0.2 Summary

0.2.1 On the novelty of results in this thesis

Before discussing the exact contents of the chapters in this thesis, I would like to address the novelty of the results. I have tried to provide references for all proofs and definitions that were not developed independently. However, I have omitted citations for the most elementary definitions and results (particularly those in the first two chapters), as I assume the reader is already familiar with them. Naturally, the material in the preliminary part of this thesis is not original. That said, some chapters in the second part contain results that, to the best of my knowledge, do not appear in the existing literature. Below, I indicate per chapter which results are novel.

To aid understanding of more technical results and proofs for monoidal categories we make extensive use of string diagrams. While challenging to typeset in LaTeX, these diagrams clarify many arguments and make proofs a lot easier to read. Proofs involving string diagrams are usually my own, but they are almost always quite straightforward.

0.2.2 Contents of the chapters

Chapter 1: General Categories

We begin by discussing *categories*, *functors*, and *natural transformations*. Categories can informally be viewed as "universes" in which certain theories reside. They consist of *objects* (e.g., groups, rings, modules, topological spaces, etc.), *morphisms* (e.g., group homomorphisms, ring homomorphisms, linear maps, continuous maps, etc.), and a *composition* on morphisms that is associative and admits a unit. It is immediately evident that this composition equips a category with a structure reminiscent of monoids, an observation that will be fundamentally important for semisimplification in later chapters. Functors can be thought of as morphisms between categories, while natural transformations serve as morphisms between functors.

Next, we briefly discuss *limits* and *colimits*, which generalise the notions of products and coproducts. We then consider categories that do not admit all limits or colimits, and particularly their completion with respect to limits or colimits. This naturally leads us to the *Yoneda* and *co-Yoneda* lemmas.

Functors that preserve limits or colimits are particularly important, and *adjoint functors* are important examples. This forms our next topic of discussion.

We conclude the chapter with a discussion on *categorification*, the process of translating set-theoretic concepts into their categorical analogues.

This chapter does not contain any new results or proofs, and some proofs will be omitted.

I would like to remark that this chapter, as well as the next three chapters, contains some material also included in a literature study I wrote last year under the supervision of Dr. Jacob C. Bridgeman [Sle24]. That study, which was about skeletal data for fusion categories, provides additional background on the categories discussed in the first three chapters, but all the necessary information for this thesis is included here.

Chapter 2: Abelian Categories

In this chapter, we discuss categories whose morphism sets carry an addition. Such categories are called *pre-additive*, and if they also admit direct sums and a null object, they are called *additive*. We then move on to additive categories that admit *kernels* and *cokernels*, which leads us to the notions of *Karoubian* and *pre-abelian categories. Abelian categories* are subsequently introduced as pre-abelian categories in which the first isomorphism theorem holds.

Short exact sequences play a fundamental role in the study of abelian categories, and they enable us to define structure-preserving functors between such categories.

We conclude the chapter with a discussion of some of the most important theorems in the theory of abelian categories: the *Jordan–Hölder* theorem, the *Krull–Schmidt* theorem, and *Schur*'s lemma.

This chapter also does not contain any new results or proofs, and some proofs will be omitted.

Chapter 3: Monoidal Categories

The third and most important chapter of the preliminary part of this thesis discusses *monoidal categories*. We begin by introducing the basic theory of monoidal categories, which are, roughly speaking, categories equipped with a bifunctor $\otimes$, called the *monoidal product* or *tensor product*, that endows the objects of the category with a monoid structure. In particular, there exists a unit for this operation, called the *monoidal unit object*. We also introduce the graphical calculus of *string diagrams* (see [Sel10]), which will be our preferred method for reasoning about monoidal categories.

Monoidal categories are designed to resemble the category of vector spaces equipped with the usual tensor product, and many definitions in the theory are motivated by this analogy. For instance, we will discuss the notions of *duals* of objects and *traces* of morphisms.

We then move on to study *braidings* on monoidal categories. A braiding is a natural isomorphism that resembles the swap map $v \otimes w \mapsto w \otimes v$ in the category of vector spaces, allowing us to permute objects in tensor products. Such a braiding endows tensor powers of objects with an action of the braid group, which factors through the symmetric group if, in addition, the braiding is *symmetric*, meaning that it is equal to its own inverse.

The monoidal product, together with a braiding, allows for generalisations of many familiar algebraic structures such as algebras and modules, which we will discuss in later chapters.

This chapter does not contain any new results, but many of the proofs involving string diagrams are my own, sometimes inspired by traditional (non-diagrammatic) arguments.

Chapter 4: Tensor Categories

In the final chapter of the preliminary part of this thesis, we discuss tensor categories. Roughly speaking, these are categories that naturally combine an abelian structure with a monoidal structure equipped with duals. This implies that monoidal product and dualisation functors should be structure-preserving as functors between abelian categories.

We begin by examining the set of endomorphisms of the monoidal unit in a monoidal category. When the category is additionally pre-additive, this set of endomorphisms becomes a ring, and every other hom-set carries the structure of a bimodule over this ring.

We then turn to categories naturally combining an abelian and a monoidal structure, leading to the notions of *multiring* and *ring categories*. Next we add duals into the mix, which results in *multitensor* and *tensor categories*.

We end the chapter by discussing symmetric tensor categories, and we explain how the action of the symmetric group on tensor powers allows us to define symmetric and exterior powers of objects. We also touch on the classification of pre-Tannakian symmetric tensor categories, as mentioned earlier. Pre-Tannakian symmetric tensor categories are such that tensor powers of objects "grow subexponentially".

Chapter 5: Semisimplification

In this chapter, we discuss *semisimplification* in considerable detail. We begin by reviewing some basic notions from the theory of *local rings*, and then show how *indecomposable* objects in abelian categories can be characterised via local rings: an object in an abelian category is indecomposable, meaning it cannot be expressed as a direct sum of two non-zero objects, if and only if its endomorphism ring is a local ring. This characterisation plays a key role in the foundational theory of ideals in abelian categories.

More generally, any pre-additive category is equipped with an addition and a composition operation, making it a categorification of a ring. This allows us to define *ideals* in pre-additive categories as categorified analogues of ring-theoretic ideals. In particular, we focus on the radical, a special ideal that resembles the Jacobson radical for rings. We show that the radical contains valuable information about whether a category is semisimple, i.e. whether every object decomposes into a direct sum of simple objects, objects that are indecomposable and have no proper subobjects.

We then turn to ideals in pre-additive categories that also carry a monoidal structure, which leads to the notion of *tensor ideals*. We demonstrate how duals enable powerful constructions of such ideals and prove the existence of a unique maximal tensor ideal, which can be built from the radical. This result illustrates an analogy between tensor categories and local rings. The chapter concludes with descriptions of the morphisms in this maximal tensor ideal, both in the framework of negligible morphisms from the literature and in a slightly more general setting.

This chapter contains some original contributions. In particular, Section 5.5.2 is entirely original (though clearly inspired by existing literature). The material in Sections 5.2-5.3 is largely based on the paper [AKO02], while Section 5.5.1 is based on [EO21a]. However, our approach to these results differs from those in the cited works; as a result, many of the statements and proofs differ (sometimes more general, sometimes less), and are "original" in the sense that they were independently developed, without directly consulting external sources (though this is not true in all cases).

Chapter 6: Algebras in Monoidal Categories

As mentioned earlier, monoidal structures on categories allow us to construct a wide range of algebraic objects. This chapter is an example of this phenomenon: we discuss *algebras* in monoidal categories, which are objects A equipped with a morphism $\mu : A \otimes A \to A$ called the *multiplication*. Our treatment is entirely general, we do not assume that algebras are *associative* or *unital*.

In general monoidal categories, that are not pre-additive, algebras are often referred to as *magmas*, and as *monoids* if they are unital and associative. If the category is pre-additive, and the monoidal product is bilinear on morphisms, then we talk about algebras and unital, associative algebras.

We also discuss modules and ideals for algebras, and explain how the action of the symmetric group on tensor powers in symmetric monoidal categories leads to a natural generalisation of Lie algebras.

The chapter concludes with a discussion of *Hopf algebras* and their *representation categories*, which give rise to symmetric tensor categories. This is a powerful construction, as demonstrated by the classification of pre-Tannakian symmetric tensor categories. We then go on to discuss affine group schemes in symmetric tensor categories.

This chapter contains one original contribution that, to our knowledge, is not found in the literature: the construction of an ideal generated by a subobject in general non-associative algebras, which can be found in Section 6.2.3.

Chapter 7: The Verlinde Category

In this very short chapter we discuss the *Verlinde category* Ver_p. We start by giving some constructions for this category as a semisimplification of a "classical" category. In particular, we extensively discuss the construction of Ver_p as the semisimplification of the representation category of the affine algebraic group α_p over an algebraically closed field of characteristic $p > 0$. We then examine the structure of tensor products in this category, and what this tells us about the tensor product in Ver_p.

We conclude by discussing subcategories of Ver_p.

This chapter does not contain any original results.

Chapter 8: Algebras in the Verlinde Category

The final chapter concerns algebras in the Verlinde category Ver_p. We examine the braiding on simple objects in this category, and explain its implications for algebra structures on these objects. We show that the only simple objects that admit an algebra structure are those of odd dimension, and that half of these algebras give rise to what we call *generalised Lie algebras*.

We also include a brief discussion on the construction of Lie algebras via semisimplification and outline some of the questions we hope to see answered in future work.

All results in this chapter are original.

Part I

Preliminaries

1 General Categories

The following four chapters cover the basic categorical tools we will need later. Almost everything written in these chapters I have personally learnt from [Lan78], [EGNO15], the nLab website ([aut25d]), some introductory sections of [Med25] written by Prof. Tom De Medts for the course on linear algebraic groups at Ghent University, a course on category theory I attended in spring 2024 taught by Dr. Ana Agore at the Vrije Universiteit Brussel (VUB, the book [Ago23] is based on the lecture notes used in this class), and a course on homological algebra I attended in autumn 2024 taught by Dr. Johanne Haugland at the Norwegian University of Science and Technology (NTNU, some lecture notes from a different lecturer are available on the course webpage). A large part of this preliminary part was inspired by a literature study I wrote under the supervision of Dr. Jacob C. Bridgeman during autumn 2023 - spring 2024 ([Sle24]), and some of the definitions were taken from there ad verbatim[1].

This first chapter will cover some basics of general category theory. For a more thorough introduction, we refer to [Lan78] or the nLab website. We do not attempt to offer much intuition behind the definitions we present here, although later sections and chapters should help develop such intuition for those encountering these ideas for the first time.

1.1 The basics

1.1.1 Categories

Naturally, the first definition in any introductory discussion of category theory should be that of a category itself.

Definition 1.1.1 (Categories). A category $\mathcal{C}$ consists of the following data:

1. A class of *objects* $\mathrm{Ob}\,(\mathcal{C})$.
2. A class of *morphisms*, $\mathrm{Hom}(\mathcal{C})$, such that every morphism f has a *domain* (or *source*) $\mathrm{dom}(f) = A \in \mathrm{Ob}\,(\mathcal{C})$, and a *codomain* (or *target*) $\mathrm{codom}(f) = B \in \mathrm{Ob}\,(\mathcal{C})$. To every two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, we assign a class of morphisms with domain A and codomain B, denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$ (the notation $\mathcal{C}(A, B)$ is quite common too). We write $f : A \to B$ or $A \xrightarrow{f} B$ to indicate that $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
3. A binary associative *composition* operation $\circ_{\mathcal{C}}$ or $\circ$ on the morphisms. Let f and g be morphisms, we call f and g *composable* if $\mathrm{codom}(f) = \mathrm{dom}(g)$[2]. The composition $g \circ f$ of these morphisms is then defined, and it is defined as a morphism with domain $\mathrm{dom}(f)$ and codomain $\mathrm{codom}(g)$. We thus have, for all $A, B \in \mathrm{Ob}(\mathcal{C})$, a map $\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C) : (f, g) \mapsto g \circ f$. We impose an associativity condition on the composition:
$$(f \circ g) \circ h = f \circ (g \circ h) \text{ for all composable } f, g, h \in \mathrm{Hom}(\mathcal{C}).$$
4. For each object $A \in \mathrm{Ob}(\mathcal{C})$, an *identity morphism* id_A (also denoted by 1_A) such that:
$$f \circ \mathrm{id}_A = f, \text{ and } \mathrm{id}_A \circ g = g,$$
if id_A, f and g, id_A are composable.

[1] Although I would like to note that in that text I adopted the convention of writing objects in categories with lowercase letters (because in that text we were studying the categories in their own regard, we were not really interested in what happens inside them, like one would be in a text on homological algebra, ...), in this text I have chosen to use uppercase letters for the objects.

[2] We say that $f_1, f_2, \ldots$ are composable if $\mathrm{codom}(f_i) = \mathrm{dom}(f_{i+1})$ for all i.

Example 1. Some of the most important examples of categories for us are

1. the category of sets, denoted $\mathbf{Set}$, where the objects are sets and the morphisms are functions between sets,
2. the category of categories, denoted $\mathbf{Cat}$, where the objects are categories and the morphisms are functors between categories (as will be defined below),
3. the category of groups, denoted $\mathbf{Grp}$, where the objects are groups and the morphisms are group morphisms,
4. the category of left (resp. right) modules over some ring R, denoted ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_R$), where the objects are left (resp. right) R-modules and the morphisms are linear morphisms between R-modules,
5. the category of abelian groups, denoted $\mathbf{Ab}$, which is just the category of left $\mathbb{Z}$-modules,
6. the category of vector spaces over some field $\mathbb{K}$, denoted $\mathbf{Vect}_{\mathbb{K}}$, which is just the category of left modules over $\mathbb{K}$,
7. the category of finite-dimensional vector spaces over some field $\mathbb{K}$, denoted $\mathbf{FinVect}_{\mathbb{K}}$.

Example 2 (The dual of a category). Let $\mathcal{C}$ be a category. The *dual* of this category[3], denoted $\mathcal{C}^{\mathrm{dual}}$, is defined as the category with the following data

1. $\mathrm{Ob}\left(\mathcal{C}^{\mathrm{dual}}\right) = \mathrm{Ob}\left(\mathcal{C}\right)$,
2. for every $f \in \mathrm{Hom}(\mathcal{C})$, we have a morphism $f^{\mathrm{dual}} \in \mathrm{Hom}(\mathcal{C}^{\mathrm{dual}})$ such that $\mathrm{dom}(f^{\mathrm{dual}}) = \mathrm{codom}(f)$ and $\mathrm{codom}(f^{\mathrm{dual}}) = \mathrm{dom}(f)$,
3. $g^{\mathrm{dual}} \circ_{\mathcal{C}^{\mathrm{dual}}} f^{\mathrm{dual}} := (f \circ_{\mathcal{C}} g)^{\mathrm{dual}}$ for all composable $f, g \in \mathrm{Hom}(\mathcal{C})$.

The dual category can be interpreted as the category where all the arrows are reversed.

It is common in category theory to speak about the *dual* of a statement. The idea is that for a class of categories closed under taking duals (such as abelian categories, see Chapter 2), a statement and its dual (obtained by formally reversing all the arrows) in this class are logically equivalent. In this text, we will frequently state both a statement and its dual within a single theorem, but prove only one of them, as the other follows by dualisation.

Example 3 (Products of categories). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We can define the *product* $\mathcal{C} \times \mathcal{D}$ as the category with the following data

1. $\mathrm{Ob}\left(\mathcal{C} \times \mathcal{D}\right) = \mathrm{Ob}\left(\mathcal{C}\right) \times \mathrm{Ob}\left(\mathcal{D}\right)$,
2. for all $A, B \in \mathrm{Ob}\left(\mathcal{C}\right)$ and $X, Y \in \mathrm{Ob}\left(\mathcal{D}\right)$, we have $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(A \times X, B \times Y) = \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and the composition is defined in the obvious way.

Example 4 (Full subcategories). Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D} \subseteq \mathcal{C}$ is a category $\mathcal{D}$ such that $\mathrm{Ob}\left(\mathcal{D}\right) \subseteq \mathrm{Ob}\left(\mathcal{C}\right)$ and $\mathrm{Hom}_{\mathcal{D}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathrm{Ob}\left(\mathcal{D}\right)$. A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called *full* if $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathrm{Ob}\left(\mathcal{D}\right)$.

There are various notions of the "size" of a category.

Definition 1.1.2. A category $\mathcal{C}$ is called

1. *locally small* if, for any two $A, B \in \mathrm{Ob}\left(\mathcal{C}\right)$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set,
2. *essentially small* if it is locally small and the class of isomorphism classes of objects is a set,
3. *small* if both $\mathrm{Ob}\left(\mathcal{C}\right)$ and $\mathrm{Hom}(\mathcal{C})$ are sets.

[3] Often also called the *opposite category*, but we will reserve the term "opposite" for monoidal categories.

Remark 1.1.3. Let $\mathcal{C}$ be any category, and let $A, B \in \mathrm{Ob}(\mathcal{C})$. Even though $\mathrm{Hom}_{\mathcal{C}}(A, B)$ might not be a set (in our definition), we will call $\mathrm{Hom}_{\mathcal{C}}(A, B)$ a *hom-set*. Many (perhaps most) authors define categories to be what we would call locally small categories. The categories we will work with in later chapters will always be locally finite (we will impose restrictions on the hom-sets that will force them to be actual sets), but we have chosen to keep this introduction quite general.

1.1.2 Morphisms

The emphasis in category theory is on morphisms rather than objects. As the above examples already show, not all morphisms are created equal, and their properties are very important to understand the structure of (and relation between) objects.

Definition 1.1.4 (Properties of morphisms). Let $\mathcal{C}$ be a category, let $A, B \in \mathrm{Ob}(\mathcal{C})$, and let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

1. f is called *epi* or an *epimorphism* if, for any $g_1, g_2 \in \mathrm{Hom}(\mathcal{C})$ such that $\mathrm{dom}(g_1) = \mathrm{dom}(g_2) = \mathrm{codom}(f)$ and $\mathrm{codom}(g_1) = \mathrm{codom}(g_2)$,
$$g_1 \circ f = g_2 \circ f \text{ implies that } g_1 = g_2. \tag{1.1}$$
f is called *split epi* or a *split epimorphism* if there exists $g : \mathrm{codom}(f) \to \mathrm{dom}(f)$ such that $f \circ g = \mathrm{id}_{\mathrm{codom}(f)}$ (that is, there exists a *section* for f).
2. f is called *mono* or a *monomorphism* if, for any $g_1, g_2 \in \mathrm{Hom}(\mathcal{C})$ such that $\mathrm{dom}(g_1) = \mathrm{dom}(g_2)$ and $\mathrm{codom}(g_1) = \mathrm{codom}(g_2) = \mathrm{dom}(f)$,
$$f \circ g_1 = f \circ g_2 \text{ implies that } g_1 = g_2. \tag{1.2}$$
f is called *split mono* or a *split monomorphism* if there exists $g : \mathrm{codom}(f) \to \mathrm{dom}(f)$ such that $g \circ f = \mathrm{id}_{\mathrm{dom}(f)}$ (that is, there exists a *retraction* for f).
3. f is called *iso* or an *isomorphism* if f is both a split epimorphism and a split epimorphism, or equivalently, if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We often write $f^{-1} := g$. Two objects are called *isomorphic* if there exists an isomorphism between them.

In analogy with $\mathrm{Hom}_{\mathcal{C}}(A, B)$, we write $\mathrm{End}_{\mathcal{C}}(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)$, $\mathrm{Iso}_{\mathcal{C}}(A, B)$ for the class of isomorphisms between two objects A and B, and $\mathrm{Aut}_{\mathcal{C}}(A) = \mathrm{Iso}_{\mathcal{C}}(A, A)$.

Monomorphisms and epimorphisms allow us to introduce subobjects and quotients in the categorical setting.

Definition 1.1.5 (Subobjects and quotients). Let $\mathcal{C}$ be a category, and let $A \in \mathrm{Ob}(\mathcal{C})$.

1. A *subobject of* A is a pair (X, f), consisting of an object $X \in \mathrm{Ob}(\mathcal{C})$, and a monomorphism $f : X \to A$. Provided with two subobjects (X, f) and (Y, g) of A, (X, f) is said to be *contained in* (Y, g) if there exists a (necessarily unique) morphism $\overline{f} : X \to Y$ such that $g \circ \overline{f} = f$.
2. A *quotient of* A is a pair (X, f), consisting of an object $X \in \mathrm{Ob}(\mathcal{C})$, and an epimorphism $f : A \to X$. Provided with two quotients (X, f) and (Y, g) of A, (Y, g) is said to be *a quotient of* (X, f) if there exists a (necessarily unique) morphism $\overline{g} : X \to Y$ such that $\overline{g} \circ f = g$.

Subobjects and quotients define categories.

Definition 1.1.6. Let $\mathcal{C}$ be a category, and let $A \in \mathrm{Ob}(\mathcal{C})$. The categories $\mathrm{Sub}(A)$ and $\mathrm{Quot}(A)$ are defined as

1. $\mathrm{Ob}\big(\mathrm{Sub}(A)\big) = \{\text{subobjects } (X, f) \text{ of } A\}$ and $\mathrm{Ob}\big(\mathrm{Quot}(A)\big) = \{\text{quotients } (X, f) \text{ of } A\}$,
2. $\mathrm{Hom}_{\mathrm{Sub}(A)}\big((X, f), (Y, g)\big) = \{\overline{f} : X \to Y \mid g \circ \overline{f} = f\}$ and $\mathrm{Hom}_{\mathrm{Quot}(A)}\big((X, f), (Y, g)\big) = \{\overline{g} : X \to Y \mid g = \overline{g} \circ f\}$.

Note that $\left|\mathrm{Hom}_{\mathrm{Sub}(A)}\big((X,f),(Y,g)\big)\right|, \left|\mathrm{Hom}_{\mathrm{Quot}(A)}\big((X,f),(Y,g)\big)\right| \leq 1$ because we are working with monomorphisms and epimorphisms respectively.

It is not hard to check that these are indeed categories.

In some categories there exist very special objects that relate to every other object in a unique way.

Definition 1.1.7 (Initial and final objects). Let $\mathcal{C}$ be a category, and let A be an object of $\mathcal{C}$. A is called *initial* (resp. *final*) if there exists exactly one morphism from (resp. to) A to (resp. from) every object of $\mathcal{C}$ (including itself). It is easy to prove that initial and final objects are unique up to unique isomorphism.

A *null object* is an object that is both initial and final. Note that every initial or final object is null once there exists one null object.

Remark 1.1.8. Let $\mathcal{C}$ be a category, let $A \in \mathrm{Ob}(\mathcal{C})$, and let $\mathcal{D} \subseteq \mathrm{Sub}(A)$ be some full subcategory; that is, a collection of subobjects of A. If $\mathcal{D}$ has an initial object, then this object is contained in all subobjects in $\mathcal{D}$. This thus implies that initial objects in categories of subobjects are "intersections" of all the subobjects in that category.

1.1.3 Functors

What makes category theory so powerful is that a wide range of mathematical structures naturally come equipped with suitable notions of morphisms, thus making category theory into a tool that can be used on this whole range of structures. It makes sense to also define a notion of morphisms between categories, effectively turning category theory into a subject that can itself be studied categorically.

Definition 1.1.9 (Functors). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{C}$ and $\mathcal{D}$ is a pair of maps

$$F_{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}) \text{ and } F_{\mathrm{Hom}} : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{D}),$$

such that for any $A, B \in \mathrm{Ob}(\mathcal{C})$ and any two composable $f, g \in \mathrm{Hom}(\mathcal{C})$:

1. $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ implies that $F_{\mathrm{Hom}}(f) \in \mathrm{Hom}_{\mathcal{D}}(F_{\mathrm{Ob}}(A), F_{\mathrm{Ob}}(B))$,
2. $F_{\mathrm{Hom}}(\mathrm{id}_A) = \mathrm{id}_{F_{\mathrm{Ob}}(A)}$ for all $A \in \mathrm{Ob}(\mathcal{C})$),
3. $F_{\mathrm{Hom}}(g \circ f) = F_{\mathrm{Hom}}(g) \circ F_{\mathrm{Hom}}(f)$.

A *contravariant functor* F between $\mathcal{C}$ and $\mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\mathrm{dual}} \to \mathcal{D}$. We will refrain from writing sentences such as "A contravariant functor $F : \mathcal{C} \to \mathcal{D} \ldots$", but we will always explicitly indicate the dual (perhaps rendering the term contravariant unnecessary).

A functor from a product of two categories is sometimes called a *bifunctor*.

Example 5 (Forgetful functors). Many categories come equipped with natural notions of so-called *forgetful functors*, these are functors that are defined by "forgetting" some of the structure of the objects. For example, many categories come with a forgetful functor to **Set** by mapping the objects (groups, modules, topological spaces, ...) to the underlying sets and the morphisms to the underlying set maps. There is also a forgetful functor from $\mathbf{Ab}$ to $\mathbf{Grp}$, or a forgetful functor from $\mathbf{Vect}_{\mathbb{K}} \to \mathbf{Ab}, \ldots$

We will denote a forgetful functor from $\mathcal{C}$ to $\mathcal{D}$ (it will always be clear what this functor should be) as $\mathrm{Forgetful}_{\mathcal{C}}^{\mathcal{D}}$.

Example 6 (Free functors). There is also a notion of *free functors* $\mathrm{Free}_{\mathcal{C}}^{\mathcal{D}}$ which map objects in the category $\mathcal{C}$ into free objects on that object in $\mathcal{D}$, this notion is linked to the notion of forgetful functors as we will see later. For example, we have the forgetful functors $\mathrm{Forgetful}_{\mathbf{Grp}}^{\mathbf{Set}}, \mathrm{Forgetful}_{\mathbf{Vect}_{\mathbb{K}}}^{\mathbf{Set}}$, and at the same time we also have free functors $\mathrm{Free}_{\mathbf{Set}}^{\mathbf{Grp}}, \mathrm{Free}_{\mathbf{Set}}^{\mathbf{Vect}_{\mathbb{K}}}$ mapping a set to the free group/vector space on that set.

Example 7 (Hom-functors). Let $\mathcal{C}$ be a locally small category, and let $A \in \mathrm{Ob}(\mathcal{C})$ be an object. We can define two *hom-functors* (a covariant and a contravariant one)

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{Set} : & B \mapsto \mathrm{Hom}_{\mathcal{C}}(A,B)\,, \text{ and} \\ & (f : B \to C) \mapsto (\mathrm{Hom}_{\mathcal{C}}(A,f) = f \circ - : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)), \end{aligned} \tag{1.3}$$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(-,A) : \mathcal{C}^{\mathrm{dual}} \to \mathbf{Set} : & B \mapsto \mathrm{Hom}_{\mathcal{C}}(B,A)\,, \text{ and} \\ & (f : B \to C) \mapsto (\mathrm{Hom}_{\mathcal{C}}(f,A) = - \circ f : \mathrm{Hom}_{\mathcal{C}}(C,A) \to \mathrm{Hom}_{\mathcal{C}}(B,A)). \end{aligned} \tag{1.4}$$

Together these combine into a hom-bifunctor

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{dual}} \times \mathcal{C} \to \mathbf{Set}. \tag{1.5}$$

For morphisms, we have already introduced the notions of monomorphisms, epimorphisms, and isomorphisms, and we can introduce similar notions for functors between categories.

Definition 1.1.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two categories $\mathcal{C}$ and $\mathcal{D}$, and let $A, B \in \mathrm{Ob}(\mathcal{C})$. F induces a map $F_{A\to B} := F_{\mathrm{Hom}}|_{\mathrm{Hom}_{\mathcal{C}}(A,B)} : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}\big(F_{\mathrm{Ob}}(A), F_{\mathrm{Ob}}(B)\big)$.

We can then define some special properties that functors can have:

1. F is called *faithful* if $F_{A\to B}$ is injective for all A, B,
2. F is called *full* if $F_{A\to B}$ is surjective for all A, B,
3. F is called *fully faithful* if $F_{A\to B}$ is bijective for all A, B,
4. F is called *dense* if every isomorphism equivalence class of $\mathcal{D}$ has a representative in the image of F_{Ob} (that is, for each $Y \in \mathrm{Ob}(\mathcal{D})$, there exists an $X \in \mathrm{Ob}(\mathcal{C})$ such that $Y \cong F(x)$),
5. F is called an *equivalence* if F is both fully faithful and dense.

Remark 1.1.11. From here on, we will write F instead of F_{Ob} and F_{Hom}.

1.1.4 Natural transformations

Continuing our journey of making category theory into something that can be studied categorically itself, it is now time to introduce morphisms between functors.

Definition 1.1.12 (Natural transformations). Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between these categories. A *natural transformation* η between F and G, denoted $\eta : F \to G$, is a class of morphisms in $\mathrm{Hom}(\mathcal{D})$

$$\eta_A : F(A) \to G(A), \text{ for all } A \in \mathrm{Ob}(\mathcal{C})\,,$$

such that the following diagram commutes for all $A, B \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle\eta_A}\downarrow & & \downarrow{\scriptstyle\eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array} . \tag{1.6}$$

Example 8 (Functor categories). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Natural transformations between functors make the class of functors from $\mathcal{C}$ to $\mathcal{D}$ into a category, showing that they are a suitable notion of morphisms of functors. We define the functor category $\mathbf{Funct}(\mathcal{C},\mathcal{D})$ as the category with the following data

1. $\mathrm{Ob}\big(\mathbf{Funct}(\mathcal{C},\mathcal{D})\big) = \{\text{functors } \mathcal{C} \to \mathcal{D}\}$,
2. for two functors $F, G : \mathcal{C} \to \mathcal{D}$, we define $\mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathcal{D})}(F,G) = \{\text{natural transformations } \mathcal{C} \to \mathcal{D}\}$, and the composition of natural transformations is defined pointwise on components.

1.2 Limits and colimits

Next, we will discuss limits and colimits. Many important constructions in mathematics are examples of limits or colimits. Some examples of explicit limit constructions that we will encounter are products and coproducts, kernels and cokernels (see Chapter 2), and the ideal of an algebra generated by a subobject (see Chapter 6).

Definition 1.2.1 (Limits and colimits). Let $\mathcal{I}, \mathcal{C}$ be two categories (usually $\mathcal{I}$ is assumed to be small, as we are usually only interested in small limits and colimits), and let $F : \mathcal{I} \to \mathcal{C}$ be a functor. A *cone* (resp. *cocone*) on this functor is a pair $(L, \{p_A : L \to F(A) \mid A \in \mathrm{Ob}\,(\mathcal{I})\})$ (resp. $(L, \{i_A : F(A) \to L \mid A \in \mathrm{Ob}\,(\mathcal{I})\})$) of an object $L \in \mathrm{Ob}\,(\mathcal{C})$, and morphisms $p_A : L \to F(A)$ (resp. $i_A : F(A) \to L$) for every $A \in \mathrm{Ob}\,(\mathcal{I})$, such that the following diagram commutes for all $A, B \in \mathrm{Ob}\,(\mathcal{I})$ and $f \in \mathrm{Hom}_{\mathcal{I}}(A, B)$

$$\begin{array}{ccc} & L & \\ {}^{p_A}\swarrow & & \searrow^{p_B} \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array} \quad \text{, resp.} \quad \begin{array}{ccc} & L & \\ {}^{i_A}\nearrow & & \nwarrow^{i_B} \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array} \quad . \tag{1.7}$$

A morphism between two cones (resp. cocones) $(C, \{r_A\}_{A \in \mathrm{Ob}(\mathcal{I})}), (D, \{s_A\}_{A \in \mathrm{Ob}(\mathcal{I})})$ is a morphism $f : C \to D$ such that $s_A \circ f = r_A$ (resp. $f \circ r_A = s_A$) for all $A \in \mathrm{Ob}\,(\mathcal{I})$.

We then obtain the category of cones (resp. cocones) on F.

1. A *limit* for F, denoted $\lim(F)$, is a final object in the category of cones on F.
2. A *colimit* for F, denoted $\mathrm{colim}(F)$, is an initial object in the category of cocones on F.

A limit (resp. colimit) is called *small* or *finite* whenever $\mathcal{I}$ is small or finite.

Example 9 (Products and coproducts). Let I be some set, and define $\mathcal{I}$ to be the discrete category on this set (that is, the category with $\mathrm{Ob}\,(\mathcal{I}) = I$, and the only morphisms are identity morphisms). Let $\mathcal{C}$ be any category, a functor $F : \mathcal{I} \to \mathcal{C}$ is then just a family of objects $\{A_i\}_{i \in I}$ indexed by I. A limit of such a functor is then called a *product* of $\{A_i\}_{i \in I}$, and is denoted $\prod_{i \in I} A_i$, and a colimit of such a functor is called a *coproduct* of $\{A_i\}_{i \in I}$, and is denoted $\coprod_{i \in I} A_i$.

Example 10 (Equalisers and coequalisers). Let $\mathcal{I}$ be the category with two objects $\star_1, \star_2$, and two non-identity morphisms $u, v : \star_1 \to \star_2$. Let $\mathcal{C}$ be any category, a functor $F : \mathcal{I} \to \mathcal{C}$ is then a choice of two objects $A, B \in \mathrm{Ob}\,(\mathcal{C})$ and two morphisms $f, g : A \to B$. A limit of such a functor is called an *equaliser* of f, g, and a colimit of such a functor is called a *coequaliser* of f, g.

Example 11 (Pullbacks and pushout). Let $\mathcal{I}$ be the category with three objects $\star_1, \star_2, \star_3$ and two morphisms $u, v : \star_1, \star_2 \to \star_3$ (resp. $u, v : \star_3 \to \star_1, \star_2$), i.e. a category that looks like this

$$\begin{array}{ccc} \star_1 & & \star_2 \\ & {}_u\searrow \quad \swarrow_v & \\ & \star_3 & \end{array} \quad \text{, resp.} \quad \begin{array}{ccc} \star_1 & & \star_2 \\ & {}_u\nwarrow \quad \nearrow_v & \\ & \star_3 & \end{array} \quad . \tag{1.8}$$

Let $\mathcal{C}$ be any category, a functor $F : \mathcal{I} \to \mathcal{C}$ is then a choice of three objects $A, B, C \in \mathrm{Ob}\,(\mathcal{C})$ and two morphisms $f, g : A, B \to C$ (resp. $f, g : C \to A, B$). A limit (resp. a colimit) of such a functor is called a *pullback* (resp. a *pushout*) of f, g.

Definition 1.2.2 (Complete and cocomplete categories). Let $\mathcal{C}$ be a category. $\mathcal{C}$ is called *complete* (resp. *cocomplete*) if all small limits (resp. colimits) exist, which means that every functor from a small category to $\mathcal{C}$ has a limit (resp. colimit).

$\mathcal{C}$ is called finitely complete (resp. finitely cocomplete) if all finite limits (resp. finite colimits) exist.

Theorem 1.2.3. *A category $\mathcal{C}$ is*

1. *finitely complete (resp. finitely cocomplete) if $\mathcal{C}$ has all equalisers and all finite products (resp. finite coproducts),*
2. *complete (resp. cocomplete) if $\mathcal{C}$ has all equalisers and all small product (resp. small coproducts).*

Proof omitted. See [Ago23, Theorem 2.4.2] or [Lan78, § V.2].

Proposition 1.2.4. *Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor that reflects isomorphisms (that is, if $F(f)$ is an isomorphism, then also f is an isomorphism).*

1. *If $\mathcal{C}$ is finitely complete (resp. finitely cocomplete), and F preserves finite limits (resp. finite colimits), then F also reflects finite limits (resp. finite colimits).*
2. *If $\mathcal{C}$ is complete (resp. cocomplete), and F preserves small limits (resp. colimits), then F also reflects small limits (resp. colimits).*

Proof omitted. See [Ago23, Proposition 2.5.9].

1.3 (Co)completions of categories and the Yoneda lemmas

In this section, we will discuss (co)completions of categories, that is their closure under taking (co)limits. In the process of doing this we will also discuss the most famous result in category theory: the Yoneda lemma.

1.3.1 Free (co)completion of categories and the Yoneda lemmas

To construct a "closure" of a category under a specified class of (co)limits, we have to find a complete category that contains the original category as a subcategory. The Yoneda lemma will provide us with the right tools to find such a category.

Theorem 1.3.1 (Yoneda lemma). *Let $\mathcal{C}$ be a locally small category, let $F : \mathcal{C} \to \mathbf{Set}$ be a functor, and let A be an object of $\mathcal{C}$. The map*

$$\xi_{A,F} : \mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(A,-),F) \to F(A) : \eta \mapsto \eta_A(\mathrm{id}_A) \tag{1.9}$$

is a natural isomorphism (in both A and F) with inverse

$$\xi_{A,F}^{-1} : F(A) \to \mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(A,-),F) : x \mapsto F(-)(x). \tag{1.10}$$

Proof. Let $B \in \mathrm{Ob}(\mathcal{C})$, let $f : A \to B$ be in $\mathcal{C}$, and let $\eta : \mathrm{Hom}_{\mathcal{C}}(A,-) \to F$ be a natural transformation. We chase the element $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ around a naturality square

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathcal{C}}(A,A) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathcal{C}}(A,B) & \quad & \mathrm{id}_A & \longmapsto & f \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} & & \downarrow & & \downarrow \\ F(A) & \xrightarrow{F(f)} & F(B) & & \eta_A(\mathrm{id}_A) & \longmapsto & \eta_B(f) \end{array} . \tag{1.11}$$

We see that $F(f)(\eta_A(\mathrm{id}_A)) = \eta_B(f)$, or thus $F(-)(\eta_A(\mathrm{id}_A)) = \eta$. The other way around, $F(\mathrm{id}_A)(x) = \mathrm{id}_{F(A)}(x) = x$, which implies that our maps are indeed inverses.

For naturality in A, let $f : A \to B$ be a morphism as in the above. The diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(A,-),F) & \xrightarrow{\xi_{A,F}} & F(A) & \quad & \eta & \longmapsto & \eta_A(\mathrm{id}_A) \\ \downarrow & & \downarrow{\scriptstyle F(f)} & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(B,-),F) & \xrightarrow{\xi_{B,F}} & F(B) & & \eta(-\circ f) & \longmapsto & F(f)(\eta_A(\mathrm{id}_A)) \end{array} \tag{1.12}$$

commutes as $\eta_B(\mathrm{id}_B \circ f) = \eta_B(f) = F(f)(\eta_A(\mathrm{id}_A))$ due to the above.

For naturality in F, let $F, G : \mathcal{C} \to \mathbf{Set}$ be two functors and let $\alpha : F \to G$ be a natural transformation between them. The diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathbf{Set})}\left(\mathrm{Hom}_{\mathcal{C}}(A,-), F\right) & \xrightarrow{\xi_{A,F}} & F(A) \\ {\scriptstyle \alpha\circ-}\downarrow & & \downarrow{\scriptstyle \alpha_A} \\ \mathrm{Hom}_{\mathbf{Funct}(\mathcal{C},\mathbf{Set})}\left(\mathrm{Hom}_{\mathcal{C}}(A,-), G\right) & \xrightarrow{\xi_{A,G}} & G(A) \end{array} \qquad \begin{array}{ccc} \eta & \longmapsto & \eta_A(\mathrm{id}_A) \\ \downarrow & & \downarrow \\ \alpha \circ \eta & \longmapsto & \alpha_A(\eta_A(\mathrm{id}_A)) \end{array} \tag{1.13}$$

commutes trivially. ■

The Yoneda lemma can be used to embed the original category into a functor category.

Definition 1.3.2 (Yoneda embedding). Let $\mathcal{C}$ be a locally small category. We define the *Yoneda embedding* as the functor

$$\mathrm{Yoneda} : \mathcal{C} \to \mathbf{Funct}(\mathcal{C}^{\mathrm{dual}}, \mathbf{Set}) : A \mapsto \mathrm{Hom}_{\mathcal{C}}(-, A) \text{ and } f \mapsto f \circ -. \tag{1.14}$$

The Yoneda embedding on $\mathcal{C}^{\mathrm{dual}}$ is sometimes also called the *contravariant Yoneda embedding*

$$\mathrm{Yoneda}_{\mathrm{contra}} : \mathcal{C}^{\mathrm{dual}} \to \mathbf{Funct}(\mathcal{C}, \mathbf{Set}) : A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, -) \text{ and } f \mapsto - \circ f. \tag{1.15}$$

Corollary 1.3.3. *Let $\mathcal{C}$ be a locally small category. The Yoneda embedding is a fully faithful functor.*

Proof. It is immediately clear that this functor is faithful as $\mathrm{Yoneda}(f)_A(\mathrm{id}_A) = f \circ \mathrm{id}_A = f$.

Let $\eta : \mathrm{Hom}_{\mathcal{C}}(-, A) \to \mathrm{Hom}_{\mathcal{C}}(-, B)$ be any natural transformation. Using the Yoneda lemma 1.3.1, we find $\eta = \mathrm{Hom}_{\mathcal{C}}(-, B)(\eta_A(\mathrm{id}_A)) = \eta_A(\mathrm{id}_A) \circ -$, which shows that η is of the form $f \circ -$ for some morphism f. ■

We will now show that this functor category is actually cocomplete, and that it is universal with this property.

Definition 1.3.4 (Free (co)completion). Let $\mathcal{C}$ be a locally small category. The Yoneda embedding

$$\mathrm{Yoneda} : \mathcal{C} \to \mathbf{Funct}(\mathcal{C}^{\mathrm{dual}}, \mathbf{Set}) \tag{1.16}$$

is called the *free cocompletion* of $\mathcal{C}$.

The *free completion* of $\mathcal{C}$ is the dual of the contravariant Yoneda embedding; that is,

$$\mathrm{Yoneda}^{\mathrm{dual}}_{\mathrm{contra}} : \mathcal{C} \to \mathbf{Funct}(\mathcal{C}, \mathbf{Set})^{\mathrm{dual}}. \tag{1.17}$$

Definition 1.3.5 (Representable functors). Let $\mathcal{C}$ be a category. A covariant (resp. contravariant) functor $F : \mathcal{C} \to \mathbf{Set}$ is called *representable* if it is naturally isomorphic to a covariant (resp. contravariant) hom-functor. Corollary 1.3.3 shows that $\mathcal{C}$ is equivalent to the full subcategory of representable functors in the free cocompletion.

Theorem 1.3.6. *Let $\mathcal{C}$ be a locally small category and let $\mathcal{D}$ be a (co)complete category. The functor category $\mathbf{Funct}(\mathcal{C}, \mathcal{D})$ is (co)complete.*

As a consequence, the free (co)completion of a locally small category is (co)complete.

Proof. We will show the statement for cocomplete categories, it is then enough to show that this category has all coproducts and coequalisers (Theorem 1.2.3). Let $(F_i)_{i\in I}$ be a family of functors $\mathcal{C} \to \mathcal{D}$, define $F := \coprod_{i\in I} F_i$ as the functor which maps objects A to $\coprod_{i\in I} F_i(A)$ and morphisms $f : A \to B$ to the unique morphism $F(f)$ (induced by the fact that $\coprod_{i\in I} F_i(A)$ is the coproduct of $(F_i(A))_{i\in I}$ in $\mathcal{D}$) making the following diagram commute for all $i \in I$

$$\begin{array}{ccc} \coprod_{i\in I} F_i(A) & \overset{\exists! F(f)}{\dashrightarrow} & \coprod_{i\in I} F_i(B) \\ {\scriptstyle \mathrm{inc}_{F_i(A)}}\uparrow & & \uparrow{\scriptstyle \mathrm{inc}_{F_i(B)}} \\ F_i(A) & \xrightarrow{F_i(f)} & F_i(B) \end{array} . \tag{1.18}$$

For $i \in I$, we define

$$\mathrm{inc}_{F_i} : F_i \to F \text{ as } (\mathrm{inc}_{F_i})_A = \mathrm{inc}_{F_i(A)} . \tag{1.19}$$

Let G be a second functor equipped with natural transformations $j_{F_i} : F_i \to G$. For $A \in \mathrm{Ob}\,(\mathcal{C})$ we then obtain a unique morphism $\eta_A : F(A) \to G(A)$ making the following diagram commute for all $i \in I$

$$\begin{array}{ccc} F(A) & \overset{\exists! \eta_A}{\dashrightarrow} & G(A) \\ {\scriptstyle \mathrm{inc}_{F_i(A)}}\uparrow & \nearrow_{(j_{F_i})_A} & \\ F_i(A) & & \end{array} . \tag{1.20}$$

These uniquely defined morphisms define a natural transformation $\eta : F \to G$ because $G(f)\circ\eta_A\circ\mathrm{inc}_{F_i(A)} = G(f) \circ (j_{F_i})_A = (j_{F_i})_B \circ F_i(f) = \eta_B \circ \mathrm{inc}_{F_i(B)} \circ F_i(f) = \eta_B \circ F(f) \circ \mathrm{inc}_{F_i(A)}$.

Similarly, one shows that the pointwise coequaliser of two natural transformations is the coequaliser of these two natural transformations. ∎

Before we can show the universal property of the free (co)completion of a category, we have to prove the co-Yoneda lemma. The proof for this theorem was adapted from this github page by Chase Meadors ([Mea]).

Theorem 1.3.7 (co-Yoneda lemma, [Lan78, § III.7, Theorem 1]). *Let $\mathcal{C}$ be a locally small category and let $F : \mathcal{C}^{\mathrm{dual}} \to$ **Set** be a (contravariant) functor. F is a (not necessarily small) colimit of representable functors.*

More explicitly, we define a category $\int F$ with

1. $\mathrm{Ob}\,(\int F) = \{(A, a) \mid A \in \mathrm{Ob}\,(\mathcal{C})\,, a \in F(A)\}$,
2. $\mathrm{Hom}_{\int F}\big((A, a), (B, b)\big) = \{f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mid F(f)(b) = a\}$.

We then have

$$F = \mathrm{colim}(\mathrm{Yoneda} \circ \pi_F), \tag{1.21}$$

where

$$\pi_F = \mathrm{Forgetful}^{\mathcal{C}}_{\int F} : \int F \to \mathcal{C} : (A, a) \mapsto A \text{ and } f \mapsto f. \tag{1.22}$$

Proof. We have to prove that F is the colimit of the functor $\mathrm{Yoneda} \circ \pi_F$, hence that there exist cocone morphisms

$$\mathrm{inc}_{(A,a)} : \mathrm{Yoneda}(\pi_F((A, a))) = \mathrm{Yoneda}(A) = \mathrm{Hom}_{\mathcal{C}}(-, A) \to F \tag{1.23}$$

that are initial in the category of cocones on $\mathrm{Yoneda} \circ \pi_F$.

We define

$$(\mathrm{inc}_{(A,a)})_B : \mathrm{Hom}_{\mathcal{C}}(B, A) \to F(B) : f \mapsto F(f)(a). \tag{1.24}$$

It is easy to check that this defines a natural transformation $\mathrm{inc}_{(A,a)}$: for $B, C \in \mathrm{Ob}\,(\mathcal{C})$ and $g : B \to C$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C, A) & \xrightarrow{(\mathrm{inc}_{(A,a)})_C} & F(C) \\ {\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathrm{Hom}_{\mathcal{C}}(B, A) & \xrightarrow{(\mathrm{inc}_{(A,a)})_B} & F(B) \end{array} \qquad \begin{array}{ccc} f & \longmapsto & F(f)(a) \\ \downarrow & & \downarrow \\ f \circ g & \longmapsto & F(f \circ g)(a) = F(g)(F(f)(a)) \end{array} . \tag{1.25}$$

The collection of natural transformations $\{\mathrm{inc}_{(A,a)} \mid (A,a) \in \mathrm{Ob}\,(\int F)\}$ defines a cocone; for morphisms $f : (A,a) \to (B,b)$ and $g : C \to A$, we have

$$((\mathrm{inc}_{(B,b)})_C \circ \mathrm{Yoneda}(f))(g) = (\mathrm{inc}_{(B,b)})_C(f \circ g) = F(f \circ g)(b) = F(g)(a) = (\mathrm{inc}_{(A,a)})_C(g). \tag{1.26}$$

Let $(G, \{j_{(A,a)}\}_{(A,a)\in\mathrm{Ob}(\int F)})$ be a second cocone on $\mathrm{Yoneda} \circ \pi_F$. We define

$$\overline{j} : F \to G \text{ through } \overline{j}_A : F(A) \to G(A) : a \mapsto (j_{(A,a)})_A(\mathrm{id}_A). \tag{1.27}$$

We will now show that $\overline{j}$ is a natural transformation $F \to G$. For $f : A \to B$, we have to prove that the following diagram commutes

$$\begin{array}{ccc} F(B) & \xrightarrow{\overline{j}_B} & G(B) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow{\overline{j}_A} & G(A) \end{array} \qquad \begin{array}{ccc} b & \longmapsto & (j_{(B,b)})_B(\mathrm{id}_B) \\ \downarrow & & \downarrow \\ F(f)(b) & \longmapsto & (j_{(A,F(f)(b))})_A(\mathrm{id}_A) = G(f)((j_{(B,b)})_B(\mathrm{id}_B)) \end{array}. \tag{1.28}$$

We know that $(j_{(B,b)})_A(f) = (j_{(A,F(f)(b))})_A(\mathrm{id}_A)$ due to the commutativity of

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(-,A) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathcal{C}}(-,B) \\ {\scriptstyle j_{(A,F(f)(b))}}\downarrow & \swarrow{\scriptstyle j_{(B,b)}} & \\ F & & \end{array} \qquad \begin{array}{ccc} \mathrm{id}_A & \longmapsto & f \\ \downarrow & \swarrow & \\ (j_{(A,F(f)(b))})_A(\mathrm{id}_A) = (j_{(B,b)})_A(f) & & \end{array}, \tag{1.29}$$

and we also know that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(B,B) & \xrightarrow{-\circ f} & \mathrm{Hom}_{\mathcal{C}}(A,B) \\ {\scriptstyle (j_{(B,b)})_B}\downarrow & & \downarrow{\scriptstyle (j_{(B,B)})_A} \\ G(B) & \xrightarrow{G(f)} & G(A) \end{array} \qquad \begin{array}{ccc} \mathrm{id}_B & \longmapsto & f \\ \downarrow & & \downarrow \\ (j_{(B,b)})_B(\mathrm{id}_B) & \longmapsto & G(f)((j_{(B,b)})_B(\mathrm{id}_B)) = (j_{(B,b)})_A(f) \end{array}. \tag{1.30}$$

It is now clear that the commutativity (1.29) and (1.30) imply the naturality of $\overline{j}$ (1.28).

The commutativity of (1.29) also shows that $\overline{j}$ makes the following diagram commute for all $(A,a) \in \mathrm{Ob}\,(\int F)$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(-,B) & & \\ {\scriptstyle \mathrm{inc}_{(B,b)}}\downarrow & \searrow{\scriptstyle j_{(B,b)}} & \\ F & \xrightarrow{\overline{j}} & G \end{array}. \tag{1.31}$$

Finally, we conclude that $\overline{j}$ is unique with this property because $a \in F(A)$ is equal to $(\mathrm{inc}_{(A,a)})_A(\mathrm{id}_A)$. ■

We are now ready to prove the universal property of the free (co)completion of a category, see this nLab page ([aut25a]).

Theorem 1.3.8 (Universal property of the free (co)completion). *Let $\mathcal{C}$ be a locally small category and let $\mathcal{D}$ be any cocomplete category equipped with a functor $F : \mathcal{C} \to \mathcal{D}$. There exists a functor (unique up to unique natural isomorphism)*

$$\overline{F} : \mathbf{Funct}(\mathcal{C}^{\mathrm{dual}}, \mathbf{Set}) \to \mathcal{D} \tag{1.32}$$

such that

1. *$\overline{F}$ preserves all colimits,*

2. *the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{Yoneda}} & \mathbf{Funct}(\mathcal{C}^{\text{dual}}, \mathbf{Set}) \\ & \searrow^{F} & \downarrow{\overline{F}} \\ & & \mathcal{D} \end{array} \quad . \tag{1.33}$$

Dually, let $\mathcal{D}$ be any complete category equipped with a functor $F : \mathcal{C} \to \mathcal{D}$. There exists a functor (unique up to natural isomorphism)

$$\overline{F} : \mathbf{Funct}(\mathcal{C}, \mathbf{Set})^{\text{dual}} \to \mathcal{D} \tag{1.34}$$

such that

1. $\overline{F}$ *preserves all limits,*
2. *the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{Yoneda}^{\text{dual}}_{\text{contra}}} & \mathbf{Funct}(\mathcal{C}, \mathbf{Set})^{\text{dual}} \\ & \searrow^{F} & \downarrow{\overline{F}} \\ & & \mathcal{D} \end{array} \quad . \tag{1.35}$$

Sketch of proof. The universal property (1.33), and the fact that the co-Yoneda lemma 1.3.7 shows that any functor in $\mathbf{Funct}(\mathcal{C}^{\text{dual}}, \mathcal{D})$ is a colimit of representable functors, suggest a definition for $\overline{F}$: for any $H : \mathcal{C}^{\text{dual}} \to \mathcal{D}$, we find a functor $\pi_H : \int H \to \mathcal{C}$ such that $H = \operatorname{colim}(\text{Yoneda} \circ \pi_H)$, we now define

$$\overline{F}(H) = \operatorname{colim}(F \circ \pi_H). \tag{1.36}$$

In fact, $\overline{F}$ is forced to be defined this way (up to natural isomorphism) because $\overline{F}$ has to preserve colimits. Let $H_1, H_2 : \mathcal{C}^{\text{dual}} \to \mathcal{D}$ be two functors, and let $\alpha : H_1 \to H_2$ be a natural transformation between them. Then α is the unique natural transformation induced by the following colimit diagram for H_1

$$\begin{array}{ccc} H_1 & \xrightarrow{\quad\alpha\quad} & H_2 \\ {\scriptstyle \text{inc}_{(A,a)} = H_1(-)(a)}\uparrow & & \uparrow{\scriptstyle \text{inc}_{(A,\alpha_A(a))} = H_2(-)(\alpha_A(a))} \\ (\text{Yoneda} \circ \pi_{H_1})(A, a) = \operatorname{Hom}_{\mathcal{C}}(-, A) & \xrightarrow{\text{id}} & (\text{Yoneda} \circ \pi_{H_2})(A, \alpha_A(a)) = \operatorname{Hom}_{\mathcal{C}}(-, A) \end{array} \quad . \tag{1.37}$$

We define $\overline{F}(\alpha)$ as the unique morphism making the following colimit diagram commute

$$\begin{array}{ccc} \overline{F}(H_1) & \xrightarrow{\exists! \overline{F}(\alpha)} & \overline{F}(H_2) \\ \uparrow & & \uparrow \\ (F \circ \pi_{H_1})(A, a) = F(A) & \xrightarrow{\text{id}_{F(A)}} & (F \circ \pi_{H_2})(A, \alpha_A(a)) = F(A) \end{array} \quad . \tag{1.38}$$

This is clearly the only possible definition of $\overline{F}$, as we can apply any candidate to the above commutative diagram (1.37).

1.3.2 Projective and inductive (co)completions of categories

In practice one often doesn't need the full free (co)completion of a category, but just the (co)completion under limits from categories that are "ordered". The reason these constructions are so important, as we will see in later chapters, is that they ensure every *filtration* has a colimit. A filtration is a chain of subobjects; that is, a sequence of monomorphisms

$$\cdots \xrightarrow{i_{n+2}} A_{n+1} \xrightarrow{i_{n+1}} A_n \xrightarrow{i_n} A_{n-1} \xrightarrow{i_{n-1}} \cdots \tag{1.39}$$

When interpreted in familiar categories such as $\mathbf{Set}$, $\mathbf{Ab}$, ${}_R\mathbf{Mod}$, this corresponds to a chain of inclusions $A_{n+1} \subseteq A_n \subseteq A_{n-1}$. To be able to apply Zorn's lemma, we want such chains to have an initial object (recall Remark 1.1.8), or equivalently, a colimit.

Definition 1.3.9 ((Co)filtered categories). Let $\mathcal{C}$ be a category.

1. $\mathcal{C}$ is called *filtered* if,

   a) for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, there exists an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $A \to C$ and $B \to C$,

   b) for any two morphisms $f, g : A \to B$, there exists an object $C \in \mathrm{Ob}(\mathcal{C})$ and a morphism $h : B \to C$ such that $h \circ f = h \circ g$.

   A colimit of a functor is called *filtered* if the domain category is filtered.

2. $\mathcal{C}$ is called *cofiltered* if,

   a) for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, there exists an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $C \to A$ and $C \to B$,

   b) for any two morphisms $f, g : A \to B$, there exists an object $C \in \mathrm{Ob}(\mathcal{C})$ and a morphism $h : C \to A$ such that $f \circ h = g \circ h$.

   A limit of a functor is called *cofiltered* if the domain category is cofiltered.

Remark 1.3.10. Filtered categories are "categorifications" (see § 1.5) of partially ordered sets.

Definition 1.3.11 (Inductive cocompletion and projective completion). Let $\mathcal{C}$ be a locally small category.

1. The *inductive cocompletion* or *ind-cocompletion*, denoted $\mathcal{C}^{\mathrm{ind}}$, is the full subcategory of the free cocompletion that consists of filtered colimits of representable functors (that is, colimits of functors from a filtered category to the full subcategory of representable functors in the free cocompletion).

2. The *projective completion* or *pro-completion*, denoted $\mathcal{C}^{\mathrm{pro}}$, is the full subcategory of the free completion that consists of cofiltered limits of representable functors (that is, limits of functors from a cofiltered category to the full subcategory of representable functors in the free completion).

Remark 1.3.12. In the literature (see, for example, [EGNO15; Ven23]), what we refer to as ind-cocompletions are often simply called ind-completions.

The inductive (resp. projective) cocompletion (resp. completion) of a category can be seen as the free filtered cocompletion (resp. completion), in analogy with the above section.

Theorem 1.3.13 (Universal property of the ind- and pro-completion). *Let $\mathcal{C}$ be a locally small category, and let $\mathcal{D}$ be any category that is cocomplete with regard to filtered functors, equipped with a functor $F : \mathcal{C} \to \mathcal{D}$. There exists a functor (unique up to unique natural isomorphism)*

$$\overline{F} : \mathcal{C}^{\mathrm{ind}} \to \mathcal{D} \tag{1.40}$$

such that

1. *$\overline{F}$ preserves all filtered colimits,*
2. *the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{Yoneda}} & \mathcal{C}^{\mathrm{ind}} \\ & \searrow^{F} & \downarrow{\scriptstyle \overline{F}} \\ & & \mathcal{D} \end{array} \; . \tag{1.41}$$

Dually, let $\mathcal{D}$ be any category that is complete with regard to filtered functors, equipped with a functor $F : \mathcal{C} \to \mathcal{D}$. There exists a functor (unique up to natural isomorphism)

$$\overline{F} : \mathcal{C}^{\mathrm{pro}} \to \mathcal{D} \tag{1.42}$$

such that

1. *$\overline{F}$ preserves all filtered limits,*
2. *the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{Yoneda}^{\text{dual}}_{\text{contra}}} & \mathcal{C}^{\text{pro}} \\ & \searrow^{F} & \downarrow{\scriptstyle \overline{F}} \\ & & \mathcal{D} \end{array} . \tag{1.43}$$

Proof omitted. The proof is analogous to the one sketched for Theorem 1.3.8.

Example 12. Let $\mathcal{C}$ be a category which has all finite coproducts and all coequalisers (hence, by Theorem 1.2.3, all finite colimits), $\mathcal{C}^{\text{ind}}$ is then the category one obtains by adding all infinite coproducts into the mix too.

For example, we have $\mathbf{FinVect}_{\mathbb{K}}^{\text{ind}} = \mathbf{Vect}_{\mathbb{K}}$. Indeed: given a vector space $V \in \text{Ob}\,(\mathbf{Vect}_{\mathbb{K}})$, define $\text{FinSub}(V)$ as the full subcategory of $\text{Sub}(V)$ consisting of all finite-dimensional subobjects. By considering spans of two subspaces, we see that this category is filtered. It is then clear that V, equipped with the inclusions, is the colimit of the canonical functor $\text{FinSub}(V) \to \mathbf{Vect}_{\mathbb{K}}$.

This shows that $\mathbf{Vect}_{\mathbb{K}}$ (or, equivalently, its category of representable functors) consists of colimits of objects in $\mathbf{FinVect}_{\mathbb{K}}$ (or equivalently its category of representable functors). As $\mathbf{Vect}_{\mathbb{K}}$ is cocomplete, we can then use the universal property of ind-cocompletions to see that $\mathbf{FinVect}_{\mathbb{K}}^{\text{ind}} = \mathbf{Vect}_{\mathbb{K}}$.

Similarly, one shows that the ind-cocompletion of the category of finitely generated groups is the category of groups, that the ind-cocompletion of finitely generated modules is the category of modules, and so forth.

Remark 1.3.14 *(Intersections in a categorical setting).* Let $\mathcal{C}$ be an arbitrary category, and let $A \in \text{Ob}\,(\mathcal{C})$. Following Remark 1.1.8, we know that collections of subobjects of A are full subcategories of $\text{Sub}(A)$. Suppose that a full subcategory $\mathcal{D} \subseteq \text{Sub}(A)$ contains (A, id_A). This category is then filtered. As a result, we know that the canonical functor $\mathcal{D} \to \mathcal{C}$ has a colimit in $\mathcal{C}^{\text{ind}}$. Let $(\cap\mathcal{D}, \{\text{inc}_X \mid (X, i_X) \in \text{Ob}\,(\mathcal{D})\})$ be the colimit for this functor. For any (X, i_X) in $\mathcal{D}$, we then know that $\text{inc}_A = i_X \circ \text{inc}_X$. As a consequence, we find that $\text{inc}_A \circ g_1 = \text{inc}_A \circ g_2$ implies that $\text{inc}_X \circ g_1 = \text{inc}_X \circ g_2$ for all X. We conclude that inc_A is a monomorphism, and hence that all colimit morphisms inc_X are monomorphisms. It is then clear that $(\cap\mathcal{D}, \text{inc}_A)$ is the intersection of $\mathcal{D}$.

We conclude that $\mathcal{C}^{\text{ind}}$ contains all intersections of collections of subobjects.

1.4 Adjoint pairs

Above we have already seen examples of forgetful functors and free functors, it turns out that these notions are linked, and can be generalised in the notion of adjoint pairs of functors.

Definition 1.4.1 (Adjoint functors). Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. (F, G) is called an *adjoint pair*, or an *adjunction*, if there exists a natural isomorphism

$$\Theta : \text{Hom}_{\mathcal{D}}\big(F(-), -\big) \to \text{Hom}_{\mathcal{C}}\big(-, G(-)\big) . \tag{1.44}$$

If (F, G) is an adjoint pair, then F is called the *left adjoint functor* to G, and G is called the *right adjoint functor* to F.

Theorem 1.4.2 (Unit and counit of adjunctions). *Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. (F, G) is an adjoint pair if and only if there exist natural transformations*

$$\eta : \text{id}_{\mathcal{C}} \to G \circ F \textit{ called the } \text{unit}, \tag{1.45}$$

$$\varepsilon : F \circ G \to \text{id}_{\mathcal{D}} \textit{ called the } \text{counit}, \tag{1.46}$$

such that for all $C \in \mathrm{Ob}(\mathcal{C}), D \in \mathrm{Ob}(\mathcal{D})$

$$\mathrm{id}_{F(C)} = \varepsilon_{F(C)} \circ F(\eta_C) \text{ and } \mathrm{id}_{G(D)} = G(\varepsilon_D) \circ \eta_{G(D)}. \tag{1.47}$$

Proof omitted. See [Ago23, Theorem 3.5.1].

Example 13 (Quasi-inverses). Let $\mathcal{C}, \mathcal{D}$ be two categories, and let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be *quasi-inverses*; that is, functors such that there exist natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F, \varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$. (F, G) and (G, F) are then two adjoint pairs.

Example 14 (Free-forgetful adjunctions). A right adjoint functor is sometimes also called a *forgetful functor*, and a left adjoint functor is then called a *free functor*. For example, let us consider the forgetful functor $\mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}} : \mathbf{Grp} \to \mathbf{Set}$ and the free functor $\mathrm{Free}^{\mathbf{Grp}}_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Grp}$. We define

$$\eta : \mathrm{id}_{\mathbf{Set}} \to \mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}} \circ \mathrm{Free}^{\mathbf{Grp}}_{\mathbf{Set}} \text{ through } \eta_X(x) = x, \tag{1.48}$$

$$\varepsilon : \mathrm{Free}^{\mathbf{Grp}}_{\mathbf{Set}} \to \mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}} \to \mathrm{id}_{\mathbf{Grp}} \text{ through } \varepsilon_G(\langle g_1, \ldots, g_n \rangle) = g_1 \cdots g_n. \tag{1.49}$$

We then find

$$\left(\varepsilon_{\mathrm{Free}^{\mathbf{Grp}}_{\mathbf{Set}}(X)} \circ \mathrm{Free}^{\mathbf{Grp}}_{\mathbf{Set}}(\eta_X)\right)(\langle x_1, \ldots, x_n \rangle) = \langle x_1, \ldots, x_n \rangle, \tag{1.50}$$

$$\left(\mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}}(\varepsilon_G) \circ \eta_{\mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}}(G)}\right)(g) = g, \tag{1.51}$$

which shows that we have an adjoint pair

$$(\mathrm{Free}^{\mathbf{Grp}}_{\mathbf{Set}}, \mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}}). \tag{1.52}$$

Theorem 1.4.3. *Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors such that (F, G) is an adjoint pair. F preserves small colimits, and G preserves small limits.*

Proof omitted. See [Ago23, Theorem 3.4.4].

1.5 Categorification

To conclude our section on general category theory, we would like to discuss categorification. Roughly speaking, this term can be taken at face value, that is: interpreting or enriching mathematical structures in categorical terms, even when they are not initially framed that way. We will discuss two kinds of categorification: horizontal and vertical categorification. Horizontal categorification can be seen as an example of the interpretation of mathematical structures in categorical terms, and vertical categorification can be seen as an example of the enrichment of mathematical structures into a categorical setting.

1.5.1 Horizontal categorification or oidification

Sometimes one realises that certain concepts in mathematics can be interpreted as categories with a single object and some additional properties. One can then construct the *oidification* or *horizontal categorification* of this concept by allowing the category to have more than one object, but still requiring the other properties to hold.

Example 15. Let us consider monoids, which we will assume to have a unit. A monoid is a set equipped with an associative binary operation that has a unit, in other words: it is a (locally small) category with one object by defining the morphisms to be the elements of the monoid and the composition to be the binary operation. The oidification of the theory of monoids is thus just the theory of (locally small) general categories[4].

[4]Typically one adds *oid* to the name of the object in question when considering the oidification (groups lead to groupoids, rings to ringoids, …), which would lead one to call categories the somewhat funny name monoidoids.

Example 16 (Groupoids). A group is just a (locally small) category with a single object in which all morphisms are invertible (i.e. isomorphisms), a *groupoid* is then any (locally small) category in which all morphisms are invertible.

Oidification often leads to the curious phenomenon that operations that were originally defined on every pair, triple, …, will suddenly only be defined partially (i.e. only on certain tuples but not on the other ones). This is because, in the one-object setting all morphisms are composable, whereas in the many-object setting composability is not guaranteed. As a result, proving (meaningful) theorems (and even finding them) about oidifications is not always straightforward.

Given a construction on some objects for which an oidification exists (and is known), the corresponding oidification of this construction can often be found by letting the same properties hold whenever they make sense.

Example 17 (Normal subgroupoids). A normal subgroup H of a group G is a subgroup $H \leq G$ such that $g^{-1}Hg \subseteq H$ for all $g \in G$. Similarly, a normal subgroupoid H of a groupoid G is a subgroupoid $H \subseteq G$ (which in the categorical setting implies that it is a subcategory) such that $g^{-1}Hg \subseteq H$ for all $g \in G$, where $g^{-1}Hg = \{g^{-1}hg \mid h \in H \text{ such that the multiplications make sense}\}$.

1.5.2 Vertical categorification

The concept of (vertical[5]) categorification is typically more involved than oidification, and we will only discuss the general idea in broad strokes here. Roughly speaking, (vertical) categorification refers to replacing set-theoretic constructions and theorems by category-theoretic analogues (that is, constructions and theorems in $\mathbf{Set}$ are lifted to constructions and theorems in $\mathbf{Cat}$). This (usually) involves lifting sets to categories, and lifting functions between sets to functors between categories satisfying some "coherence" or "compatibility" conditions (we will see examples of these in Chapter 3) enforced by natural isomorphisms (the idea behind these is that they replace equalities).

The following Table 1.1, taken from This Week's Finds in Mathematical Physics (Week 121) by John Baez ([Bae98]), shows what common set-theoretic concepts translate to after categorification:

$\mathbf{Set}$	$\mathbf{Cat}$
Sets	Categories
Elements of sets	Objects of categories
Equations between elements	Isomorphisms between objects
Functions	Functors
Equations between functions	Natural isomorphisms between functors

Table 1.1: The dictionary of categorification: it allows us to prepare a cookbook of recipes in $\mathbf{Cat}$ from the cookbook of recipes in $\mathbf{Set}$.

Contrary to oidification, it is often much harder to construct categorifications of certain well-known objects or constructions (it is usually quite hard to find the correct coherence conditions and prove some coherence theorem). However, once one has a coherence theorem, the theory of the objects or constructions translates to the categorification in a straightforward way.

[5]We will sometimes omit the word vertical when considering vertical categorification, as we will refer to horizontal categorification as oidification.

2 Abelian Categories

In the first chapter we discussed general categories. We are now ready to start looking at specific flavours of categories, eventually combining them in the very flavourful world of symmetric tensor categories.

We begin with perhaps the most well-known flavour: abelian categories. In [Lan78, Chapter VIII], abelian categories are introduced as categories satisfying a set of axioms that "suffice to prove all the facts about commuting diagrams and connecting morphisms which are proved in $\mathbf{Ab}$ by methods of chasing elements."

Many details will be omitted in this section, as we assume that the reader is already familiar with the material presented here. We will present the main definitions, but will not discuss most theorems and proofs (these can be found in any text on homological algebra, and are also extensively discussed in my earlier literature study [Sle24]).

Everything in this chapter can be found in [Lan78] or [EGNO15].

2.1 Pre-additive and additive categories

2.1.1 Pre-additive categories

One straightforward way to make categories more flavourful is by *enriching* the hom-sets. Enriched categories have a formal definition, but we will not need this definition in this text (they are, however, discussed in [Sle24]). For our purposes, it suffices to know that a category $\mathcal{C}$ enriched over another (monoidal, see Chapter 3) category $\mathcal{D}$ is one in which the hom-sets of $\mathcal{C}$ are actually objects of $\mathcal{D}$, and composition is given by a morphism in $\mathcal{D}$, meaning that it respects the structure of $\mathcal{D}$.

Definition 2.1.1 (Pre-additive categories). A *pre-additive category*, or $\mathbf{Ab}$-*enriched category*, is a category $\mathcal{C}$ in which each hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ (for $A, B \in \mathrm{Ob}(\mathcal{C})$) is equipped with the structure of an abelian group $(\mathrm{Hom}_{\mathcal{C}}(A, B), +)$, and composition of morphisms is bilinear; that is,

$$\begin{aligned} f \circ (g + h) &= f \circ g + f \circ h \text{ if } f, g \text{ and } f, h \text{ are composable,} \\ (f + g) \circ h &= f \circ h + g \circ h \text{ if } f, h \text{ and } g, h \text{ are composable.} \end{aligned} \tag{2.1}$$

More generally, provided with a commutative ring R, a category is called ${}_R\mathbf{Mod}$-*enriched* if every hom-set is equipped with a left R-module structure, and the composition is bilinear. In this setting, hom-sets are also referred to as *hom-spaces*. If all hom-spaces are finitely generated as a left R-module, then the category is called ${}_R\mathbf{FinMod}$-*enriched*.

As always, pre-additive categories come with structure-preserving functors.

Definition 2.1.2 (Additive functors). Let $\mathcal{C}$ and $\mathcal{D}$ be pre-additive categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *additive* if, for any two $A, B \in \mathrm{Ob}(\mathcal{C})$, $F_{A\to B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}\big(F(A), F(B)\big)$ is a group morphism.

2.1.2 Additive categories

In pre-additive categories, we can generalise the notion of direct sums on abelian groups, modules, …

Definition 2.1.3 (Biproducts or direct sums). Let $\mathcal{C}$ be a pre-additive category, and let $A_1, \ldots, A_n \in \mathrm{Ob}(\mathcal{C})$. A *biproduct* or *direct sum* of $A_1, \ldots, A_n$ is an object $A_1 \oplus \cdots \oplus A_n$, together with morphisms $\mathrm{inc}_{A_k} : A_k \to A_1 \oplus \cdots \oplus A_n$ and $\mathrm{proj}_{A_k} : A_1 \oplus \cdots \oplus A_n \to A_k$ for all k such that the following identities hold

$$\mathrm{id}_{A_k} = \mathrm{proj}_{A_k} \circ \mathrm{inc}_{A_k} \quad \text{for all } k, \tag{2.2}$$
$$\mathrm{id}_{A_1 \oplus \cdots \oplus A_n} = \mathrm{inc}_{A_1} \circ \mathrm{proj}_{A_1} + \cdots + \mathrm{inc}_{A_n} \circ \mathrm{proj}_{A_n}, \tag{2.3}$$
$$0 = \mathrm{proj}_{A_\ell} \circ \mathrm{inc}_{A_k} \quad \text{if } k \neq \ell. \tag{2.4}$$

Remark 2.1.4. For two objects, the last identity (2.4) is unnecessary. Indeed, (2.2) and (2.3) imply that

$$\begin{aligned} \mathrm{proj}_{A_2} \circ \mathrm{inc}_{A_1} &= \mathrm{proj}_{A_2} \circ (\mathrm{inc}_{A_1} \circ \mathrm{proj}_{A_1} + \mathrm{inc}_{A_2} \circ \mathrm{proj}_{A_2}) \circ \mathrm{inc}_{A_1} \\ &= \mathrm{proj}_{A_2} \circ \mathrm{inc}_{A_1} \circ \mathrm{id}_{A_1} + \mathrm{id}_{A_2} \circ \mathrm{proj}_{A_2} \circ \mathrm{inc}_{A_1} \\ &= \mathrm{proj}_{A_2} \circ \mathrm{inc}_{A_1} + \mathrm{proj}_{A_2} \circ \mathrm{inc}_{A_1} \end{aligned} . \tag{2.5}$$

Remark 2.1.5. The term “biproduct” stems from the fact that two objects in a pre-additive category have a product or coproduct if and only if they have a biproduct (see [Lan78, § VIII.2, Theorem 2]).

Definition 2.1.6 (Additive categories). Let $\mathcal{C}$ be a pre-additive category. $\mathcal{C}$ is called *additive* if it has a null object 0 (i.e. an object that is both initial and final), and all binary biproducts exist.

Provided with a commutative ring R, a category $\mathcal{C}$ is called *R-linear* if it is ${}_R\mathbf{Mod}$-enriched and additive.

Lemma 2.1.7. *Let $\mathcal{C}$ be a pre-additive category with a null object Z. For any $A, B \in \mathrm{Ob}(\mathcal{C})$, let $z_A^B : A \to B$ be the unique morphism factoring through Z, and let $0_A^B : A \to B$ be the zero for the addition on $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Then always $z_A^B = 0_A^B$.*

Proof. Let $f : B \to C$ be any morphism, then $f \circ 0_A^B = 0_A^C$ as $f \circ 0_A^B + f \circ 0_A^B = f \circ 0_A^B$. As a consequence, we find $z_A^B = z_B^B \circ 0_A^B = 0_A^B$. ■

From now on, we will always denote both the zero element for the addition *and* the morphism factoring through the null object by 0.

Remark 2.1.8. One would expect that there is a notion of pre-additive functors for pre-additive categories and a separate notion of additive functors (which would be pre-additive and, in addition, preserve the biproducts) for additive categories. However, it turns out that these two notions are equivalent. Indeed, it is straightforward to verify that any functor which is linear on morphisms preserves biproducts; that is, it preserves identities (2.2), (2.3), and (2.4).

Definition 2.1.9 (Simple, semisimple, and indecomposable objects). Let $\mathcal{C}$ be an additive category. An object $A \in \mathrm{Ob}(\mathcal{C})$ is called

1. *simple* if 0 and A are its only subobjects, i.e. if for any monomorphism $i : B \to A$ we have either $B \cong 0$ or $B \cong A$,
2. *semisimple* if it is a finite biproduct of simple objects,
3. *indecomposable* if it does not admit a decomposition into a biproduct of its subobjects, or equivalently if it is not isomorphic to a biproduct of two non-zero objects.

The category $\mathcal{C}$ is called *semisimple* if every object is semisimple.

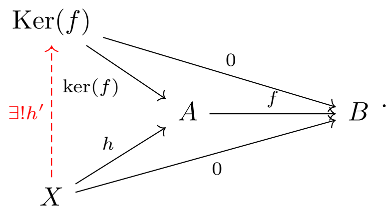
2.2 Karoubian, pre-abelian, and abelian categories

To pass from additive categories to abelian categories we will introduce the categorical generalisation of kernels.

2.2.1 Kernels and cokernels

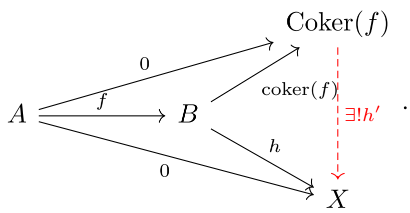
Definition 2.2.1 (Kernel and cokernel of a morphism). Let $\mathcal{C}$ be a category with a null object. A *kernel* of a morphism $f : A \to B$ is an equaliser of the morphisms $f, 0 : A \to B$.

This means that a kernel of a morphism $f : A \to B$ is a pair $(\mathrm{Ker}(f), \ker(f))$, consisting of an object $\mathrm{Ker}(f) \in \mathrm{Ob}\,(\mathcal{C})$ and a morphism $\ker(f) : \mathrm{Ker}(f) \to A$, such that $f \circ \ker(f) = 0$, and such that every morphism $h : X \to A$ satisfying $f \circ h = 0$ factors uniquely through $\ker(f)$; that is, there exists a unique morphism $h' : X \to \mathrm{Ker}(f)$ such that $h = \ker(f) \circ h'$



(2.6)

A *cokernel* is the dual of the above construction; a coequaliser of the morphisms $f, 0 : A \to B$. This means that a cokernel of a morphism $f : A \to B$ is a pair $(\mathrm{Coker}(f), \mathrm{coker}(f))$, consisting of an object $\mathrm{Coker}(f) \in \mathrm{Ob}\,(\mathcal{C})$ and a morphism $\mathrm{coker}(f) : B \to \mathrm{Coker}(f)$, such that $\mathrm{coker}(f) \circ f = 0$, and such that every morphism $h : B \to X$ satisfying $h \circ f = 0$ factors uniquely through $\mathrm{coker}(f)$; that is, there exists a unique morphism $h' : \mathrm{Coker}(f) \to X$ such that $h = h' \circ \mathrm{coker}(f)$



(2.7)

Through the standard proofs for limits and colimits, it is easy to prove that kernels and cokernels are unique up to unique isomorphism.

Definition 2.2.2 (Image and coimage of a morphism). Let $\mathcal{C}$ be a category with a null object, and let f be a morphism.

1. If f admits a cokernel $(\mathrm{Coker}(f), \mathrm{coker}(f))$, such that this cokernel admits a kernel, then this kernel of the cokernel is called an *image* of f

$$(\mathrm{Im}(f), \mathrm{im}(f)) = (\mathrm{Ker}(\mathrm{coker}(f)), \ker(\mathrm{coker}(f))). \tag{2.8}$$

2. If f admits a kernel $(\mathrm{Ker}(f), \ker(f))$, such that this kernel admits a cokernel, then this cokernel of the kernel is called a *coimage* of f

$$(\mathrm{Coim}(f), \mathrm{coim}(f)) = (\mathrm{Coker}(\ker(f)), \mathrm{coker}(\ker(f))). \tag{2.9}$$

Lemma 2.2.3. *Let $\mathcal{C}$ be a category with a null object 0. If a morphism $f : A \to B$ in $\mathcal{C}$ admits a kernel* $(\mathrm{Ker}(f), \ker(f))$ *(resp. a cokernel* $(\mathrm{Coker}(f), \mathrm{coker}(f))$*), then* $\ker(f)$ *is a monomorphism (resp.* $\mathrm{coker}(f)$ *is an epimorphism).*

Proof. Suppose that $k := \ker(f) \circ g_1 = \ker(f) \circ g_2$ for two morphisms g_1, g_2. We then have $f \circ k = 0$, which implies that there exists a unique morphism $\overline{g}$ such that $\ker(f) \circ \overline{g} = k$. Clearly, $\overline{g} = g_1 = g_2$. ■

Proposition 2.2.4. *Let $\mathcal{C}$ be a pre-additive category with a null object 0. A morphism $f : A \to B$ with a kernel $(\mathrm{Ker}(f), \ker(f))$ (resp. a cokernel $(\mathrm{Coker}(f), \mathrm{coker}(f))$) is a monomorphism (resp. epimorphism) if and only if $\mathrm{Ker}(f) \cong 0$ and $\ker(f) = 0$ (resp. $\mathrm{Coker}(f) \cong 0$ and $\mathrm{coker}(f) = 0$).*

Proof. Suppose that f is a monomorphism. Then $f \circ \ker(f) = f \circ 0$ implies that $\ker(f) = 0$, hence also that $\mathrm{Ker}(f) \cong 0$ as kernels are monomorphisms and the null morphism 0 can only be a monomorphism if the domain is a null object. Suppose now that f is such that $\mathrm{Ker}(f) \cong 0$ and $\ker(f) = 0$. If $f \circ g_1 = f \circ g_2 = 0$, then $f \circ (g_1 - g_2) = 0$, which implies that there is a unique morphism $\overline{g}$ such that $g_1 - g_2 = \ker(f) \circ \overline{g} = 0$, hence $g_1 = g_2$. ■

2.2.2 Karoubian categories

The above definition of kernels allows us to show that biproducts and idempotent endomorphisms are essentially the same thing.

Proposition 2.2.5. *Let $\mathcal{C}$ be a pre-additive category with a null object 0, and let $A \in \mathrm{Ob}(\mathcal{C})$. The following are equivalent*

1. *there exists a non-trivial (i.e. not 0 or id_A) idempotent endomorphism $f : A \to A$ such that f and $\mathrm{id}_A - f$ admit a kernel,*
2. *there exists a non-trivial idempotent morphism $f : A \to A$ such that f and $\mathrm{id}_A - f$ admit a cokernel,*
3. *there exists a non-trivial object (i.e. not isomorphic to A or 0) $X \in \mathrm{Ob}(\mathcal{C})$ together with a split monomorphism $f : X \to A$ that admits a cokernel,*
4. *there exists a non-trivial object $X \in \mathrm{Ob}(\mathcal{C})$ together with a split epimorphism $f : A \to X$ that admits a kernel,*
5. *A is decomposable (i.e. not indecomposable).*

Proof. Assume that (1) holds. Set $X := \mathrm{Ker}(f), Y := \mathrm{Ker}(\mathrm{id}_A - f)$ and $\mathrm{inc}_X := \ker(f), \mathrm{inc}_Y := \ker(\mathrm{id}_A - f)$. As $f \circ (\mathrm{id}_A - f) = 0$, there is a uniquely induced morphism $\mathrm{proj}_X : A \to X$ such that $\mathrm{inc}_X \circ \mathrm{proj}_X = \mathrm{id}_A - f$. Similarly, there is a unique morphism $\mathrm{proj}_Y : A \to Y$ such that $\mathrm{inc}_Y \circ \mathrm{proj}_Y = f$. We then find $\mathrm{inc}_X \circ \mathrm{proj}_X \circ \mathrm{inc}_X = \mathrm{inc}_X$, which implies that $\mathrm{proj}_X \circ \mathrm{inc}_X = \mathrm{id}_X$ because inc_X is a monomorphism by Lemma 2.2.3. Similarly, we show that $\mathrm{proj}_Y \circ \mathrm{inc}_Y = \mathrm{id}_Y$. Because $f, \mathrm{id}_A - f \neq 0$, we know that X and Y are not null objects. Indeed, this would imply that f or $\mathrm{id}_A - f$ is a monomorphism, and thus that $f = \mathrm{id}_A$ or $\mathrm{id}_A - f = \mathrm{id}_A$ (and thus $\mathrm{id}_A - f = 0$ or $f = 0$) through $f \circ f = f$ and $(\mathrm{id}_A - f) \circ (\mathrm{id}_A - f) = \mathrm{id}_A - f$. This proves that $A = X \oplus Y$ is decomposable, and we conclude that (1) implies (5).

Similarly, we prove that (2) implies (5).

Suppose now that (5) holds, and let $A = X \oplus Y$ be a biproduct decomposition into non-zero objects, with biproduct morphisms $\mathrm{inc}_X : X \to A, \mathrm{proj}_X : A \to X$. Set $f := \mathrm{inc}_X \circ \mathrm{proj}_X$. f is idempotent through $\mathrm{proj}_X \circ \mathrm{inc}_X = \mathrm{id}_X$, and as X and Y are not null objects, we find $f \neq 0, \mathrm{id}_A$. It is not hard to see that f and $\mathrm{id}_A - f$ admit kernels and cokernels and that inc_X and proj_X admit a cokernel and a kernel respectively: the kernels of f and proj_X coincide, and they equal inc_Y as $f \circ k = 0$ implies that $k = (\mathrm{inc}_X \circ \mathrm{proj}_X + \mathrm{inc}_Y \circ \mathrm{proj}_Y) \circ k = \mathrm{inc}_Y \circ (\mathrm{proj}_Y \circ k)$ (and this induced morphism $\mathrm{proj}_Y \circ k$ is unique as inc_Y is a monomorphism). We conclude that (5) implies (1), (2), (3), and (4).

Finally, we show that (3) implies (2) and that (4) implies (1): for any split monomorphism (resp. epimorphism) $f : X \to A$ (resp. $f : A \to X$) with splitting $\overline{f} : A \to X$ (resp. $\overline{f} : X \to A$), we find that the composition $f \circ \overline{f}$ (resp. $\overline{f} \circ f$) is an idempotent endomorphism with the same cokernel (resp. cokernel) as f. ■

Corollary 2.2.6. *Let $\mathcal{C}$ be a pre-additive category with a null object. The following are equivalent*

1. *every idempotent endomorphism in $\mathcal{C}$ admits a kernel,*

2. *every idempotent endomorphism in $\mathcal{C}$ admits a cokernel,*
3. *for every idempotent endomorphism $f : A \to A$ in $\mathcal{C}$, there exists a biproduct decomposition $A = X \oplus Y$ such that $f = \mathrm{inc}_X \circ \mathrm{proj}_X$.*

Proof. This follows from the statement and proof of the above Proposition 2.2.5. ■

The above characterisation of biproducts through idempotent endomorphisms motivates the following type of pre-additive categories.

Definition 2.2.7 (Karoubian categories). A pre-additive category $\mathcal{C}$ with a null object is called *Karoubian* if it satisfies one of the equivalent properties in Corollary 2.2.6.

Remark 2.2.8. We call an idempotent endomorphism $f : A \to A$ *split* if there exists an object B and morphisms $r : A \to B$, $s : B \to A$ such that $s \circ r = f$ and $r \circ s = \mathrm{id}_B$. Note that s is then a split monomorphism and r a split epimorphism. The above statements show that idempotents that admit kernels (or cokernels) split. Furthermore, the splitting morphisms r and s then admit a kernel and cokernel respectively.

In particular, every idempotent morphism in a Karoubian category splits, and every split monomorphism and epimorphism admit a cokernel and kernel respectively.

2.2.3 Abelian categories

Definition 2.2.9 (Pre-abelian categories). Let $\mathcal{C}$ be an additive category. $\mathcal{C}$ is called *pre-abelian* if every morphism admits a kernel and a cokernel.

Corollary 2.2.10. *Let $\mathcal{C}$ be a pre-abelian category, and let $A \in \mathrm{Ob}\,(\mathcal{C})$ be simple. Any morphism from (resp. to) A is either zero or a monomorphism (resp. an epimorphism).*

Proof. Let $f : A \to X$ be a morphism. Since $\ker(f) : \mathrm{Ker}(f) \to A$ is a monomorphism, we are in one of two cases: either $\ker(f) = 0$, in which case $\mathrm{Ker}(f) = 0$ and f is a monomorphism by Proposition 2.2.4, or $\ker(f)$ is an isomorphism, in which case $f = 0$. ■

We are finally ready to introduce the most important categories in homological algebra: abelian categories.

Definition 2.2.11 (Abelian categories). Let $\mathcal{C}$ be a pre-abelian category, and let $f : A \to B$ be a morphism in $\mathcal{C}$. For every kernel and cokernel of f, there exists a unique morphism $\overline{f}$ making the following diagram commute

$$\begin{array}{ccccccc} \mathrm{Ker}(f) & \xrightarrow{\ker(f)} & A & \xrightarrow{f} & B & \xrightarrow{\mathrm{coker}(f)} & \mathrm{Coker}(f) \\ & & {\scriptstyle \mathrm{coim}(f)}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{im}(f)} & & \\ & & \mathrm{Coim}(f) & \xrightarrow{\exists ! \overline{f}} & \mathrm{Im}(f) & & \end{array} . \tag{2.10}$$

$\mathcal{C}$ is called *abelian* if this induced morphism $\overline{f}$ is always an isomorphism.

Remark 2.2.12. Alternatively, a pre-abelian category $\mathcal{C}$ is called abelian if every monomorphism is a kernel, and every epimorphism is a cokernel.

Remark 2.2.13. Because $\mathrm{Coim}(f)$ plays the role of $A/\,\mathrm{Ker}(f)$, we see that a pre-abelian category is abelian precisely when it satisfies the first isomorphism theorem.

Proposition 2.2.14. *Let $\mathcal{C}$ be an abelian category. A morphism in $\mathcal{C}$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

Proof. Isomorphisms are always split monomorphisms and epimorphisms, so we only have to prove the other direction. If f is both a monomorphism and an epimorphism, then Proposition 2.2.4 implies that $\ker(f) = 0$ and $\mathrm{coker}(f)$. As a consequence, we find $\mathrm{coim}(f) = \mathrm{id}_A$ and $\mathrm{im}(f) = \mathrm{id}_B$. The definition of abelian categories now implies that $f = \overline{f}$ is an isomorphism. ■

Example 18 (Module categories). The standard examples of abelian categories (and in some sense the only ones) are categories of left or right modules over some ring. Let R be a ring, the category of left R-modules ${}_R\mathbf{Mod}$ is clearly enriched over itself (hence $\mathbf{Ab}$-enriched), has a zero object (the zero module), admits all kernels and cokernels, admits all finite biproducts, and is such that the first isomorphism theorem holds. The category of finitely generated left (resp. right) R-modules is denoted ${}_R\mathbf{FinMod}$ (resp. $\mathbf{FinMod}_R$), and is abelian too.

Example 19 (Representation categories). Let G be a group and let $\mathbb{K}$ be some field. The category of $\mathbb{K}$-linear representations $\mathbf{Rep}_{\mathbb{K}}(G)$ is additively isomorphic to the category of left modules over its group algebra $\mathbb{K}G$ (i.e. $\mathbf{Rep}_{\mathbb{K}}(G) \cong {}_{\mathbb{K}G}\mathbf{Mod}$), and is thus abelian too. The category of finite-dimensional representations $\mathbf{FinRep}_{\mathbb{K}}(G)$ is abelian too and isomorphic to ${}_{\mathbb{K}G}\mathbf{FinMod}$.

2.3 Short exact sequences and exact functors

2.3.1 Short exact sequences

Next, we introduce one of the most important tools in homological algebras: (short) exact sequences.

Definition 2.3.1 (Exact sequences). Let $\mathcal{C}$ be an abelian category. A sequence of morphisms in $\mathcal{C}$

$$\cdots \xrightarrow{f_{i-2}} A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots \tag{2.11}$$

is called a *chain complex* if $f_k \circ f_{k-1} = 0$ for all k, and is called *exact* at degree i if $(\mathrm{Im}(f_{i-1}), \mathrm{im}(f_{i-1})) = (\mathrm{Ker}(f_i), \ker(f_i))$.

A sequence of morphisms is called

1. *exact* if it is exact in every degree,
2. *left exact* if it is exact and of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \ , \tag{2.12}$$

   note that this just means that f is a kernel of g,

3. *right exact* if it is exact and of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \ , \tag{2.13}$$

   note that this just means that g is a cokernel of f,

4. *short exact* if it is exact and of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \ . \tag{2.14}$$

Example 20. Let $\mathcal{C}$ be an abelian category and let $A, B \in \mathrm{Ob}(\mathcal{C})$. We can define two special short exact sequences coming from the biproduct $A \oplus B$

$$\begin{gathered} 0 \longrightarrow A \xrightarrow{\mathrm{inc}_A} A \oplus B \xrightarrow{\mathrm{proj}_B} B \longrightarrow 0 \\ . \\ 0 \longrightarrow B \xrightarrow{\mathrm{inc}_B} A \oplus B \xrightarrow{\mathrm{proj}_A} A \longrightarrow 0 \end{gathered} \tag{2.15}$$

2.3.2 Exact functors

Exact sequences allow us to elegantly introduce the structure-preserving functors for abelian categories.

Definition 2.3.2 (Exact functors). Let $\mathcal{C}, \mathcal{D}$ be abelian categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is called

1. *left exact* if it preserves left exact sequences, i.e. maps kernel to kernels,
2. *right exact* if it preserves right exact sequences, i.e. maps cokernels to cokernels,
3. *exact* if it preserves short exact sequences.

Remark 2.3.3 *(Exact functors are the correct structure-preserving functors for abelian categories).* It is easy to prove that a functor is exact if and only if it is both left and right exact.

Example 21 (Hom-functors). Let $\mathcal{C}$ be an abelian category, and let $A \in \mathrm{Ob}(\mathcal{C})$ be an object. The covariant and contravariant hom-functors $\mathrm{Hom}_{\mathcal{C}}(A, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, A)$ are left exact. Indeed, let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \tag{2.16}$$

be exact.

Then

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{C}}(A, Y) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{C}}(A, Z) \tag{2.17}$$

is exact: $f \circ -$ is a monomorphism as f is a monomorphism, and $g \circ h = 0$ implies that there exists a unique $\overline{h}$ such that $h = f \circ \overline{h}$ (as $f = \ker(g)$).

Similarly, one can prove that $\mathrm{Hom}_{\mathcal{C}}(-, A)$ maps right exact sequences to left exact sequences.

2.3.3 Split short exact sequences

Above, we have shown that every biproduct induces a short exact sequence. We can characterise these short exact sequences through our discussion on Karoubian categories.

Corollary 2.3.4. *Let $\mathcal{C}$ be an abelian category, and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.18}$$

be a short exact sequence in $\mathcal{C}$.

The following are equivalent

1. *f is a split monomorphism,*
2. *g is a split epimorphism,*
3. *$B = A \oplus C$ with inclusion $f : A \to B$ and projection $g : B \to C$.*

Proof. This follows from Proposition 2.2.5. ■

Definition 2.3.5 (Split short exact sequences). Let $\mathcal{C}$ be an abelian category. A short exact sequence is called *split short exact* if one of the equivalent properties of Corollary 2.3.4 holds.

2.3.4 Projective objects

In module theory, projective modules play a very important role. Similarly, projective objects in abelian categories are very important.

Definition 2.3.6 (Projective and injective objects). Let $\mathcal{C}$ be an abelian category. An object $A \in \mathrm{Ob}(\mathcal{C})$ is called *projective* if

$$\mathrm{Hom}_{\mathcal{C}}(A, -) \text{ is exact (or, equivalently due to Example 21, right exact).} \tag{2.19}$$

Equivalently, A is projective if, for any $f : A \to X$ and $g : X \to Y$ such that $X \xrightarrow{g} Y \to 0$ is exact, there is an induced morphism $\overline{f} : A \to X$ making the following diagram commute

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\exists \overline{f}} & \downarrow f & & \\ X & \xrightarrow{g} & Y & \longrightarrow & 0 \end{array} \quad . \tag{2.20}$$

Dually, A is called *injective* if

$$\mathrm{Hom}_{\mathcal{C}}(-, A) \text{ is exact (or, equivalently, right exact).} \tag{2.21}$$

Equivalently, A is injective if, for any $f : X \to A$ and $g : X \to Y$ such that $0 \to X \xrightarrow{g} Y$ is exact, there is an induced morphism $\overline{f} : Y \to A$ making the following diagram commute

$$\begin{array}{ccccc} & & A & & \\ & & f \uparrow & \nwarrow^{\exists \overline{f}} & \\ 0 & \longrightarrow & X & \xrightarrow{g} & Y \end{array} \quad . \tag{2.22}$$

Remark 2.3.7. Projective and injective objects can also be defined for additive categories through (2.20) and (2.22).

Lemma 2.3.8. *Let $\mathcal{C}$ be an additive category, and let $A \in \mathrm{Ob}(\mathcal{C})$ be projective (resp. injective). Any epimorphism to A (resp. monomorphism from A) is split.*

Proof. Let $g : X \to A$ be an epimorphism. We then find $\overline{\mathrm{id}_A}$ such that $g \circ \overline{\mathrm{id}_A} = \mathrm{id}_A$. ■

The following proposition is quite remarkable, as it shows that the most interesting additive categories contain objects that are not projective.

Proposition 2.3.9. *Let $\mathcal{C}$ be a Karoubian category. Every object of $\mathcal{C}$ is projective if and only if every epimorphism is split.*

Proof. If every object is projective, then every epimorphism is split through Lemma 2.3.8.

Suppose now that every monomorphism is split. Let $A, X, Y \in \mathrm{Ob}(\mathcal{C})$ be any objects, let $f : A \to X$ be any morphism, and let $g : X \to Y$ be any epimorphism. g splits, which implies that there exists $\overline{g}$ such that $g \circ \overline{g} = \mathrm{id}_Y$. We can then set $\overline{f} = \overline{g} \circ f$, and we have $g \circ \overline{f} = f$. ■

2.4 The Jordan-Hölder theorem and Schur's lemma

Now that we have introduced the basic definitions of abelian categories, we can state some of the most important results (at least for our purposes) that hold in this setting.

2.4.1 The Jordan-Hölder theorem

Definition 2.4.1 (Filtrations and Jordan-Hölder series). Let $\mathcal{C}$ be an abelian category. A *filtration* of an object $A \in \mathrm{Ob}\,(\mathcal{C})$ is a finite sequence of monomorphisms

$$0 = A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} \dots \xrightarrow{i_{n-2}} A_{n-1} \xrightarrow{i_{n-1}} A_n = A \ . \tag{2.23}$$

A filtration is called a *Jordan-Hölder series* if, for every k, the object $A_{k+1}/A_k = \mathrm{Coker}(i_k)$ is simple. If such a filtration exists, then A is called *of finite length*. The length of this filtration is then called the *length* of A, denoted $\mathrm{len}(A)$.

Let $X \in \mathrm{Ob}\,(\mathcal{C})$ be a simple object, the *multiplicity* of A in the Jordan-Hölder series of A, denoted $[A : X]$, is defined as the amount of times X is isomorphic to the object $A_{k+1}/A_k = \mathrm{Coker}(i_k)$.

Definition 2.4.2. Let $\mathcal{C}$ be any category, and let $A \in \mathrm{Ob}\,(\mathcal{C})$.

A is called *artinian* if it satisfies the descending chain condition on subobjects, i.e. if every descending chain of subobjects

$$\cdots \xrightarrow{i_3} A_2 \xrightarrow{i_2} A_1 \xrightarrow{i_1} A_0 = A \tag{2.24}$$

eventually becomes stationary. This means that there exists some $n \geq 0$ such that i_k is an isomorphism for all $k \geq n$.

A is called *noetherian* if it satisfies the ascending chain condition.

$\mathcal{C}$ is called artinian or noetherian if every object is artinian or noetherian.

Example 22. Semisimple abelian categories are artinian, noetherian, and such that all objects are of finite length.

Remark 2.4.3. One can show that an object is of finite length if and only if it is both artinian and noetherian, we refer to the Stacks Project page on Jordan-Hölder ([aut25e]).

Theorem 2.4.4 (Krull-Schmidt, [EGNO15, Theorem 1.5.7]). *Let $\mathcal{C}$ be an abelian category, and let A be an object in $\mathcal{C}$ of finite length. A is isomorphic to a biproduct of indecomposable objects, and this decomposition is unique up to isomorphism. By this last statement, we mean that for two decompositions $A \cong B_1 \oplus \cdots \oplus B_m \cong C_1 \oplus \cdots \oplus C_n$ into indecomposables, we have $m = n$, and there exists a permutation σ such that $B_i \cong C_{\sigma(i)}$ for all i.*

Proof omitted. This is a corollary of [Kra, Theorem 4.2] and Corollary 5.1.10 (which we will prove later). ◻

Definition 2.4.5 (Krull-Schmidt categories). Let $\mathcal{C}$ be an additive category. $\mathcal{C}$ is called *Krull-Schmidt* if the above theorem holds for all objects in $\mathcal{C}$; that is, if every object $A \in \mathrm{Ob}\,(\mathcal{C})$ has a unique (up to isomorphism) biproduct decomposition into indecomposable objects.

Theorem 2.4.6 (Jordan-Hölder, [EGNO15, Theorem 1.5.4]). *Let $\mathcal{C}$ be an abelian category, and let $A \in \mathrm{Ob}\,(\mathcal{C})$ be an object of finite length. Any filtration of A can be extended to a Jordan-Hölder series of A, any two Jordan-Hölder series of A have the same length, and the multiplicities of simple objects in any two Jordan-Hölder series are the same.*

This last statement can be restated in the following way: for any two Jordan-Hölder series $0 = A_0, A'_0 \to \cdots \to A_m, A'_n = A$, we have $m = n$, and there exists a permutation σ such that $A_{k+1}/A_k \cong A'_{\sigma(k)+1}/A'_{\sigma(k)}$ for all k.

Proof omitted. We refer to the Stacks Project page on Jordan-Hölder ([aut25e]). ◻

We can use this result to prove that semisimple categories can be recognised through short exact sequences splitting.

Proposition 2.4.7. *An abelian category in which all objects are of finite length is semisimple if and only if every short exact sequence splits.*

Proof. It is easy to see that every short exact sequence in a semisimple abelian category splits.

Suppose now that $\mathcal{C}$ is such that every short exact sequence splits. Let X be any object of $\mathcal{C}$, and let Y be the first non-zero object in a Jordan-Hölder series for X. The short exact sequence induced by the inclusion of Y into X splits, which shows that $X \cong Y \oplus X'$ for some object X'. It is then easy to see that X' is an object such that $\mathrm{len}(X') = \mathrm{len}(X) - 1$, and we see that X is semisimple through induction. ■

Corollary 2.4.8. *For an abelian category $\mathcal{C}$ in which all objects are of finite length, the following are equivalent*

1. *$\mathcal{C}$ is semisimple,*
2. *every monomorphism in $\mathcal{C}$ is split,*
3. *every epimorphism in $\mathcal{C}$ is split,*
4. *every object in $\mathcal{C}$ is projective.*

Proof. The statements (1) and (2) or (3) are equivalent through Proposition 2.4.7, and (3) and (4) are equivalent through Proposition 2.3.9. ■

2.4.2 Schur's lemma

Theorem 2.4.9 (Schur's lemma, [EGNO15, Lemma 1.5.2]). *Let $\mathcal{C}$ be an abelian category, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be simple objects. Any non-zero morphism $A \to B$ is an isomorphism.*

Proof omitted. It is not hard to prove this by using the properties of simple objects, and the fact that a morphism in an abelian category is an isomorphism if and only if it is both a monomorphism and an epimorphism (Proposition 2.2.14).

Corollary 2.4.10 ([EGNO15, Proposition 1.8.4]). *Let $\mathcal{C}$ be a $\mathbf{FinVect}_{\mathbb{K}}$-enriched abelian category, with $\mathbb{K}$ some algebraically closed field, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be simple. We have*

$$\dim_{\mathbb{K}}(\mathrm{Hom}_{\mathcal{C}}(A, B)) = \begin{cases} 1 & \text{if } A \cong B \\ 0 & \text{else} \end{cases}. \tag{2.25}$$

Proof omitted. One can prove this statement by noting that Schur's lemma 2.4.9 implies that the hom-spaces will be finite-dimensional division algebras over $\mathbb{K}$.

Definition 2.4.11 (Schur categories). An additive category $\mathcal{C}$ is called *Schur* if it satisfies Schur's lemma; that is, if for any two simple objects $A, B \in \mathrm{Ob}(\mathcal{C})$ the hom-space $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a division ring (skew field).

3 Monoidal Categories

In this chapter, we introduce one of the most important categorical structures used throughout this text: *monoidal products*.

Monoidal categories, that is, categories equipped with a monoidal product, can be approached from two intuitive points of view:

1. A monoidal category can be viewed as a (vertical) categorification of a monoid. The collection of objects $\mathrm{Ob}\,(\mathcal{C})$ carries a monoid-like structure, with a binary multiplication $\otimes$ and a unit object for this multiplication $\mathbb{1}$. This "multiplication" is promoted to a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the monoidal product. This is the viewpoint adopted in [EGNO15, § 2.1].
2. In [Lan78, § VII.1], monoidal categories are introduced as categories that "come equipped with a "product" like the direct product $\times$, the direct sum $\oplus$, or the tensor product $\otimes$". This is a viewpoint that will be particularly useful throughout this text. Starting from Chapter 4, we will focus on monoidal products that are bilinear on morphisms. This implies that the monoidal product does not behave like a direct product $\times$ or a direct sum $\oplus$ (indeed, it is not hard to check that these notions are not bilinear in the usual settings), but rather like a tensor product $\otimes$ of modules, vector spaces, or representations.

In Chapter 6, we will see that monoidal products allow us to define algebras in categories.

3.1 The basics

3.1.1 Monoidal categories

Definition 3.1.1 (Monoidal categories, [Lan78, § VII.1] and [EGNO15, Definition 2.1.1 and Definition 2.2.8]). A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consists of

- a category $\mathcal{C}$,
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (we write $\star_1 \otimes \star_2 = \otimes(\star_1, \star_2)$ for objects and morphisms), called the *monoidal product*,
- an object $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$, called the *monoidal unit*,

and three natural isomorphisms:

- $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, called the *associator*, this is a natural isomorphism between the functors

$$F : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C} : (A, B, C) \mapsto (A \otimes B) \otimes C \text{ and } (f, g, h) \mapsto (f \otimes g) \otimes h, \text{ and}$$
$$G : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C} : (A, B, C) \mapsto A \otimes (B \otimes C) \text{ and } (f, g, h) \mapsto f \otimes (g \otimes h),$$

implying that diagrams of the form

$$\begin{array}{ccc} (A_1 \otimes B_1) \otimes C_1 & \xrightarrow{(f \otimes g) \otimes h} & (A_2 \otimes B_2) \otimes C_2 \\ {\scriptstyle \alpha_{(A_1, B_1, C_1)}} \downarrow & & \downarrow {\scriptstyle \alpha_{(A_2, B_2, C_2)}} \\ A_1 \otimes (B_1 \otimes C_1) & \xrightarrow{f \otimes (g \otimes h)} & A_2 \otimes (B_2 \otimes C_2) \end{array} \tag{3.1}$$

commute,

- $\lambda : \mathbb{1} \otimes - \to \mathrm{id}_{\mathcal{C}}$, called the *left unitor*, this is a natural isomorphism between the functors

$$\begin{aligned} &L : \mathcal{C} \to \mathcal{C} : A \mapsto \mathbb{1} \otimes A \text{ and } f \mapsto \mathrm{id}_{\mathbb{1}} \otimes f, \text{ and} \\ &\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} : A \mapsto A \text{ and } f \mapsto f, \end{aligned}$$

  implying that diagrams of the form

$$\begin{array}{ccc} \mathbb{1} \otimes A & \xrightarrow{\mathrm{id}_{\mathbb{1}} \otimes f} & \mathbb{1} \otimes B \\ {\scriptstyle \lambda_A}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \tag{3.2}$$

  commute,

- $\rho : - \otimes \mathbb{1} \to \mathrm{id}_{\mathcal{C}}$, called the *right unitor*, this is a natural isomorphism between the functors

$$\begin{aligned} &R : \mathcal{C} \to \mathcal{C} : A \mapsto A \otimes \mathbb{1} \text{ and } f \mapsto f \otimes \mathrm{id}_{\mathbb{1}}, \text{ and} \\ &\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} : A \mapsto A \text{ and } f \mapsto f, \end{aligned}$$

  implying that diagrams of the form

$$\begin{array}{ccc} A \otimes \mathbb{1} & \xrightarrow{f \otimes \mathrm{id}_{\mathbb{1}}} & B \otimes \mathbb{1} \\ {\scriptstyle \rho_A}\big\downarrow & & \big\downarrow{\scriptstyle \rho_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \tag{3.3}$$

  commute.

These natural isomorphisms have to be such that the following properties hold:

1. $\alpha_{(A,B,C)} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is a natural isomorphism such that

$$\begin{array}{ccccc} & & ((A \otimes B) \otimes C) \otimes D & & \\ & \swarrow{\scriptstyle \alpha_{(A,B,C)} \otimes \mathrm{id}_D} & & {\scriptstyle \alpha_{(A \otimes B,C,D)}}\searrow & \\ (A \otimes (B \otimes C)) \otimes D & & & & (A \otimes B) \otimes (C \otimes D) \\ {\scriptstyle \alpha_{(A,B \otimes C,D)}}\big\downarrow & & & & \big\downarrow{\scriptstyle \alpha_{(A,B,C \otimes D)}} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow{\qquad \mathrm{id}_A \otimes \alpha_{(B,C,D)} \qquad} & & A \otimes (B \otimes (C \otimes D)) \end{array} \tag{3.4}$$

   commutes. This identity is called the *pentagon identity*.

2. $\lambda_A : \mathbb{1} \otimes A \to A$ and $\rho_A : A \otimes \mathbb{1} \to A$ are natural isomorphisms such that

$$\begin{array}{ccc} (A \otimes \mathbb{1}) \otimes B & \xrightarrow{\qquad \alpha_{(A,\mathbb{1},B)} \qquad} & A \otimes (\mathbb{1} \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array} \tag{3.5}$$

   commutes. This identity is called the *triangle identity*.

3. $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$, the unitors coincide on the monoidal unit.

We will often use the shorthand notation $\mathcal{C}$ for the monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$.

Examples

Example 23 (The opposite of a monoidal category). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. We can define a monoidal category $(\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1}, \alpha^{\mathrm{op}}, \lambda^{\mathrm{op}}, \rho^{\mathrm{op}})$, called the *opposite* monoidal category[1], through

1. $A \otimes^{\mathrm{op}} B := B \otimes A$,
2. $\alpha^{\mathrm{op}}_{(A,B,C)} := \alpha^{-1}_{(C,B,A)} : (A \otimes^{\mathrm{op}}) \otimes^{\mathrm{op}} C = C \otimes (B \otimes A) \to A \otimes^{\mathrm{op}} (B \otimes^{\mathrm{op}} C) = (C \otimes B) \otimes A$,
3. $\lambda^{\mathrm{op}}_A := \rho_A : \mathbb{1} \otimes^{\mathrm{op}} A = A \otimes \mathbb{1} \to A$, and $\rho^{\mathrm{op}}_A := \lambda_A : A \otimes^{\mathrm{op}} \mathbb{1} = \mathbb{1} \otimes A \to A$.

Example 24 (Vector spaces). Let $\mathbb{K}$ be a field. The standard example of a monoidal category is the category of vector spaces $\mathbf{Vect}_{\mathbb{K}}$, where the monoidal product is the standard tensor product $\otimes_{\mathbb{K}}$, and the monoidal unit is $\mathbb{K}$.

Example 25 (Representations). Let G be a group and let $\mathbb{K}$ be a field. The category of $\mathbb{K}$-linear G-representations $\mathbf{Rep}_{\mathbb{K}}(G)$ equipped with the tensor product induced by the one on $\mathbf{Vect}_{\mathbb{K}}$ (i.e. $(V, \rho) \otimes (W, \sigma) = (V \otimes_{\mathbb{K}} W, \rho \otimes \sigma)$) defines a monoidal category with unit $(\mathbb{K}, g \mapsto \mathrm{id}_{\mathbb{K}})$.

Example 26 (Modules). Let R be a commutative ring. The category of R-modules ${}_R\mathbf{Mod}$, equipped with the usual tensor product $\otimes_R$ and monoidal unit $\mathbb{1} = R$, forms a monoidal category.

The above examples are all examples of monoidal categories for which the monoidal product is a "tensor product", now we will introduce some examples in which the monoidal product does not feel like a tensor product because of a lack of bilinearity.

Example 27 (Sets). $\mathbf{Set}$, equipped with the Cartesian product of sets and $\mathbb{1} = \{\star\}$, is a monoidal category.

Example 28 (Additive categories). Any additive category with zero object 0 is a monoidal category when equipped with the biproduct $\oplus$ as the monoidal product and $\mathbb{1} = 0$ as the unit.

3.1.2 Monoidal functors and natural transformations

As always, it is possible to define structure-preserving functors on monoidal categories.

Definition 3.1.2 (Monoidal functors, [EGNO15, Definition 2.4.1 and Definition 2.4.5]). Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha_{\mathcal{C}}, \lambda_{\mathcal{C}}, \rho_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}})$ be monoidal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *monoidal* if there exists an isomorphism $\zeta : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$, and there exists a natural isomorphism $\varepsilon_{(A,B)} : F(A) \otimes_{\mathcal{D}} F(B) \to F(A \otimes_{\mathcal{C}} B)$, such that

$$\begin{array}{ccccc}
(F(A) \otimes_{\mathcal{D}} F(B)) \otimes_{\mathcal{D}} F(C) & \xrightarrow{\varepsilon_{(A,B)} \otimes_{\mathcal{D}} \mathrm{id}_{F(C)}} & F(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{D}} F(C) & \xrightarrow{\varepsilon_{(A \otimes_{\mathcal{C}} B, C)}} & F((A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C) \\
{\scriptstyle (\alpha_{\mathcal{D}})_{(F(A),F(B),F(C))}} \downarrow & & & & \downarrow {\scriptstyle F((\alpha_{\mathcal{C}})_{(A,B,C)})} \\
F(A) \otimes_{\mathcal{D}} (F(B) \otimes_{\mathcal{D}} F(C)) & \xrightarrow{\mathrm{id}_{F(A)} \otimes_{\mathcal{D}} \varepsilon_{(B,C)}} & F(A) \otimes_{\mathcal{D}} F(B \otimes_{\mathcal{C}} C) & \xrightarrow{\varepsilon_{(A, B \otimes_{\mathcal{C}} C)}} & F(A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C))
\end{array}, \tag{3.6}$$

and

$$\begin{array}{ccc}
\mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(A) & \xrightarrow{(\lambda_{\mathcal{D}})_{F(A)}} & F(A) \\
{\scriptstyle \zeta \otimes_{\mathcal{D}} \mathrm{id}_{F(A)}} \uparrow & & \uparrow {\scriptstyle F((\lambda_{\mathcal{C}})_A)} \\
F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(A) & \xrightarrow{\varepsilon_{(\mathbb{1}_{\mathcal{C}}, A)}} & F(\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A)
\end{array}
\qquad
\begin{array}{ccc}
F(A) \otimes_{\mathcal{D}} \mathbb{1}_{\mathcal{D}} & \xrightarrow{(\rho_{\mathcal{D}})_{F(A)}} & F(A) \\
{\scriptstyle \mathrm{id}_{F(A)} \otimes_{\mathcal{D}} \zeta} \uparrow & & \uparrow {\scriptstyle F((\rho_{\mathcal{C}})_A)} \\
F(A) \otimes_{\mathcal{D}} F(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{\varepsilon_{(A, \mathbb{1}_{\mathcal{C}})}} & F(A \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}})
\end{array}. \tag{3.7}$$

commute for all objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$. F is called a *monoidal equivalence* if F is, in addition, an equivalence of categories $\mathcal{C} \to \mathcal{D}$.

As monoidal functors come with some additional data, we also need a notion of structure-preserving natural transformations.

[1] Also called the *reverse monoidal category*, but we will reserve the term "reverse" for braided monoidal categories.

Definition 3.1.3 (Monoidal natural transformations, [EGNO15, Definition 2.4.8]). Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha_{\mathcal{C}}, \lambda_{\mathcal{C}}, \rho_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}})$ be monoidal categories, and let $(F, \varepsilon, \zeta), (\overline{F}, \overline{\varepsilon}, \overline{\zeta}) : \mathcal{C} \to \mathcal{D}$ be monoidal functors. A *monoidal natural transformation* $\eta : (F, \varepsilon, \zeta) \to (\overline{F}, \overline{\varepsilon}, \overline{\zeta})$ is a natural transformation $\eta : F \to \overline{F}$ such that the following diagram commutes for all $A, B \in \mathrm{Ob}(\mathcal{C})$

$$\begin{array}{ccc} F(A) \otimes_{\mathcal{C}} F(B) & \xrightarrow{\varepsilon(A,B)} & F(A \otimes B) \\ {\scriptstyle \eta_A \otimes \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_{A \otimes B}} \\ \overline{F}(A) \otimes_{\mathcal{C}} \overline{F}(B) & \xrightarrow{\overline{\varepsilon}(A,B)} & \overline{F}(A \otimes B) \end{array} . \tag{3.8}$$

Remark 3.1.4 *([EGNO15, Remark 2.4.10]).* One can show that any monoidal equivalence has a monoidal quasi-inverse.

Remark 3.1.5. The above definition of monoidal categories (respectively, monoidal functors) may initially appear quite involved, but it is in fact the natural categorification of the concept of monoids (respectively, morphisms of monoids). Indeed, the *Mac Lane Coherence Theorem* (see [Lan78, § VII.2] and [EGNO15, Theorem 2.9.2]) shows that the coherence conditions (3.4) and (3.5) guarantee that the theory of monoidal categories is a categorification of the theory of monoids (that is, general results about monoids lift to corresponding results about monoidal categories).

3.2 Mac Lane's strictness theorem

We have mentioned that, even though the definition of monoidal categories in Definition 3.1.1 may seem rather involved, it is conceptually satisfying due to its interpretation as a categorification of the theory of monoids. However, there is a variant of the notion of monoidal categories that is significantly easier to work with in theory[2].

Definition 3.2.1 (Strict monoidal categories, [Lan78, § VII.1] and [EGNO15, Definition 2.8.1]). A *strict monoidal category* is a a monoidal category with a trivial associator and trivial unitors. Explicitly, a strict monoidal category is a triple $(\mathcal{C}, \otimes, \mathbb{1})$ of a category $\mathcal{C}$, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and an object $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$, such that for all objects A, B, C and all morphisms f, g, h in $\mathcal{C}$

1. the monoidal product makes $\mathrm{Ob}(\mathcal{C})$ into a monoid, i.e. the monoidal product is associative on objects $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, and $\mathbb{1}$ acts as a unit for this product $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
2. this monoid structure also holds functorially, i.e. $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and $\mathrm{id}_{\mathbb{1}} \otimes f = f = f \otimes \mathrm{id}_{\mathbb{1}}$.

Remarkably, every monoidal category is equivalent to a strict monoidal category.

Theorem 3.2.2 (Mac Lane's strictness theorem, [Lan78, § XI.3] and [EGNO15, Theorem 2.8.5]). *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Proof omitted. See [Lan78, Theorem 1, § XI.3]. ☒

This theorem shows that, up to equivalence, it is safe to work with strict monoidal categories instead of general monoidal categories.

[2]We would like to note that this notion of strict monoidal categories is easier to work with in *theory*, indeed: when working with monoidal categories in more practical settings (for example in condensed matter physics or quantum computing), researchers often prefer to work with skeletal monoidal categories (especially for fusion categories, where through *skeletal data* fusion categories can be stored as a finite set of data). Not all monoidal categories are equivalent to categories that are both strict *and* skeletal (see [Lan78, § VII.1])!

3.3 String diagrams

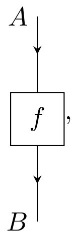
The strict monoidal categories we defined above allow us to introduce a particularly elegant graphical tool: *string diagrams*. Also known as the *graphical calculus* of monoidal categories, string diagrams generalise similar visual languages found throughout mathematics and physics. Appearing, for instance, in 2-category theory[3] (see [Lan78, § XII.3]), Penrose notation in physics, quantum computing, topological quantum field theory, knot theory, and more.

Our discussion of string diagrams is based on this GitHub page by Jutho Haegeman ([Hae]). For a more thorough introduction to the various graphical languages for monoidal categories, we refer to [Sel10].

As we have emphasised in the above, morphisms are the central focus in category theory. It is therefore natural that a diagrammatic language used for categorical reasoning should place morphisms at the heart. In string diagrams the edges correspond to objects (or, equivalently, their identity morphisms), and the vertices correspond to morphisms.

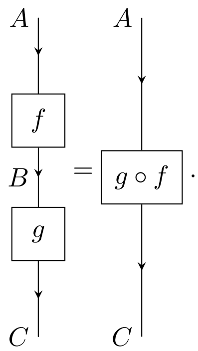
Suppose that we have some strict monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$.

A morphism $f : A \to B$ in $\mathcal{C}$ is drawn as



(3.9)

and composition with a second morphism $g : B \to C$ in $\mathcal{C}$ is denoted by concatenation of the diagrams

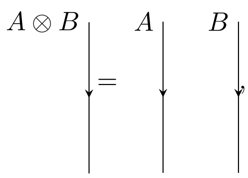


(3.10)

The identity morphism is never drawn, and is thus just an edge.

We will often omit the labels of the objects, as these are usually clear from context.

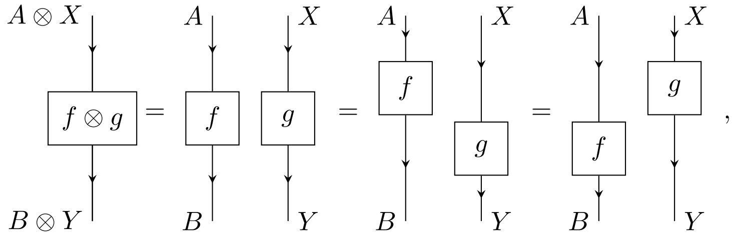
A diagrammatic language for monoidal categories should, of course, incorporate monoidal products. Parallel edges represent monoidal products. For example



(3.11)

[3]A monoidal category can be viewed as a 2-category with a single object.

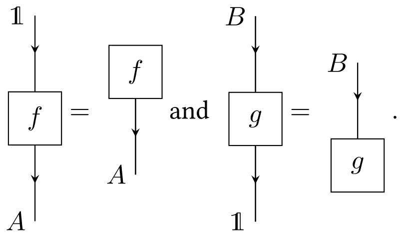
and for morphisms $f : A \to B, g : X \to Y$



(3.12)

where the last two equalities follow from the functoriality of the monoidal product.

Usually, the monoidal unit $\mathbb{1}$ is omitted in string diagrams, for example



(3.13)

3.4 Duals

3.4.1 Left and right duals

We are now ready to start introducing more interesting properties (or additional data) that a monoidal category can have, starting with duals which generalise the notion of duals for finitely generated projective modules, finite-dimensional vector spaces, …

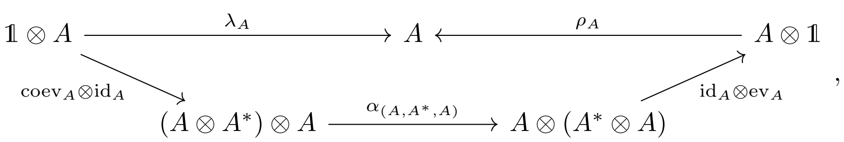
Definition 3.4.1 (Left and right duals, [EGNO15, Definition 2.10.1 and Definition 2.10.2]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category, and let A be an object.

1. A *left dual* of A is a triple $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$ of an object $A^* \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\mathrm{ev}_A : A^* \otimes A \to \mathbb{1}$ (called the *evaluation*) and $\mathrm{coev}_A : \mathbb{1} \to A \otimes A^*$ (called the *coevaluation*), such that the following compositions are the identity morphisms:

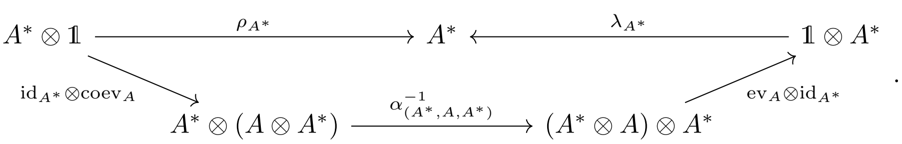
$$A \xrightarrow{\lambda_A^{-1}} \mathbb{1} \otimes A \xrightarrow{\mathrm{coev}_A \otimes \mathrm{id}_A} (A \otimes A^*) \otimes A \xrightarrow{\alpha_{(A,A^*,A)}} A \otimes (A^* \otimes A) \xrightarrow{\mathrm{id}_A \otimes \mathrm{ev}_A} A \otimes \mathbb{1} \xrightarrow{\rho_A} A,$$

$$A^* \xrightarrow{\rho_{A^*}^{-1}} A^* \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{A^*} \otimes \mathrm{coev}_A} A^* \otimes (A \otimes A^*) \xrightarrow{\alpha^{-1}_{(A^*,A,A^*)}} (A^* \otimes A) \otimes A^* \xrightarrow{\mathrm{ev}_A \otimes \mathrm{id}_{A^*}} \mathbb{1} \otimes A^* \xrightarrow{\lambda_{A^*}} A^*.$$

Equivalently, the following diagrams commute

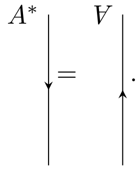


(3.14)



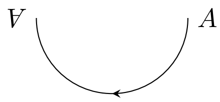
(3.15)

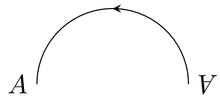
In string diagram notation, duals are represented by rotating the diagrams upside down, for example



(3.16)

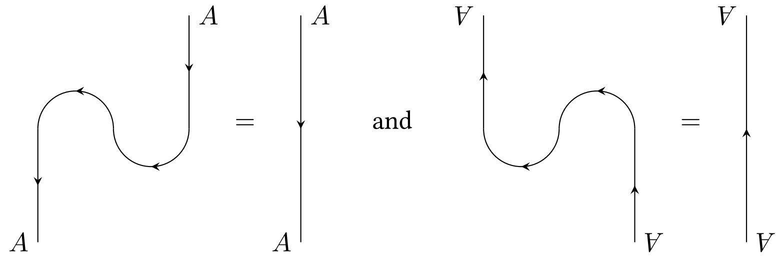
Evaluation and coevaluation morphisms will be drawn as arcs, without writing out the morphisms explicitly, i.e.

$\mathrm{ev}_A =$ 

and $\mathrm{coev}_A =$ 

. (3.17)

With the aid of this graphical calculus, we can then rewrite the above identities (often called the *zigzag* or *snake identities*) in the following way



. (3.18)

2. A *right dual* of A is a triple $({}^*A, \mathrm{ev}_A, \mathrm{coev}_A)$ of an object ${}^*A \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\mathrm{ev}_A : A \otimes {}^*A \to \mathbb{1}$ and $\mathrm{coev}_A : \mathbb{1} \to {}^*A \otimes A$, such that the following compositions are the identity morphisms:

$$A \xrightarrow{\rho_A^{-1}} A \otimes \mathbb{1} \xrightarrow{\mathrm{id}_A \otimes \mathrm{coev}_A} A \otimes ({}^*A \otimes A) \xrightarrow{\alpha^{-1}_{(A,{}^*A,A)}} (A \otimes {}^*A) \otimes A \xrightarrow{\mathrm{ev}_A \otimes \mathrm{id}_A} \mathbb{1} \otimes A \xrightarrow{\lambda_A} A,$$

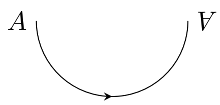
$${}^*A \xrightarrow{\lambda^{-1}_{{}^*A}} \mathbb{1} \otimes {}^*A \xrightarrow{\mathrm{coev}_A \otimes \mathrm{id}_{{}^*A}} ({}^*A \otimes A) \otimes {}^*A \xrightarrow{\alpha_{({}^*A,A,{}^*A)}} {}^*A \otimes (A \otimes {}^*A) \xrightarrow{\mathrm{id}_{{}^*A} \otimes \mathrm{ev}_A} {}^*A \otimes \mathbb{1} \xrightarrow{\rho_{{}^*A}} {}^*A.$$

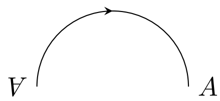
Equivalently, the following diagrams commute

$$\begin{array}{ccccc} A \otimes \mathbb{1} & \xrightarrow{\rho_A} & A & \xleftarrow{\lambda_A} & \mathbb{1} \otimes A \\ & \searrow^{\mathrm{id}_A \otimes \mathrm{coev}_A} & & {}^{\mathrm{ev}_A \otimes \mathrm{id}_A}\nearrow & \\ & A \otimes ({}^*A \otimes A) & \xrightarrow{\alpha^{-1}_{(A,{}^*A,A)}} & (A \otimes {}^*A) \otimes A & \end{array} , \qquad (3.19)$$

$$\begin{array}{ccccc} \mathbb{1} \otimes {}^*A & \xrightarrow{\lambda_{{}^*A}} & {}^*A & \xleftarrow{\rho_{{}^*A}} & {}^*A \otimes \mathbb{1} \\ & \searrow^{\mathrm{coev}_A \otimes \mathrm{id}_{{}^*A}} & & {}^{\mathrm{id}_{{}^*A} \otimes \mathrm{ev}_A}\nearrow & \\ & ({}^*A \otimes A) \otimes {}^*A & \xrightarrow{\alpha_{({}^*A,A,{}^*A)}} & {}^*A \otimes (A \otimes {}^*A) & \end{array} . \qquad (3.20)$$

Graphically, not a lot changes for right duals. The only difference is that the arrows on the arcs point in the opposite direction

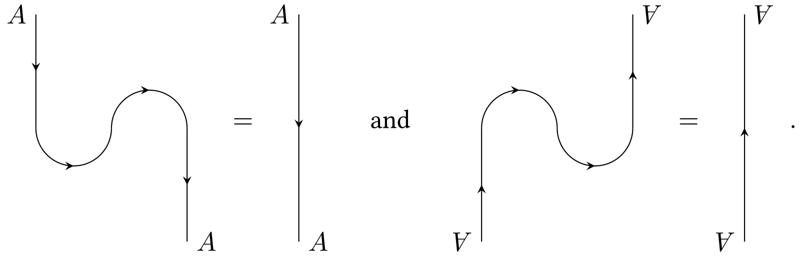
$\mathrm{ev}_A =$ 

and $\mathrm{coev}_A =$ 

. (3.21)

Graphically one can thus deduce whether we are working with left or right duals by looking at the direction of the arrows.

The zigzag identities for right duals then become



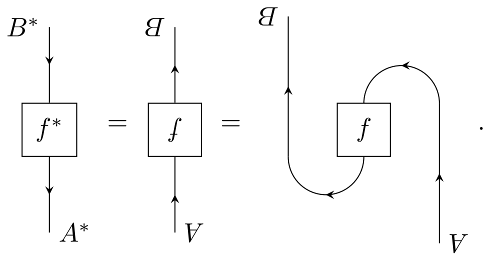
(3.22)

Let A, B be objects of $\mathcal{C}$, and let $f : A \to B$ be a morphism.

1. Suppose that $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$ is a left dual of A, and that $(B^*, \mathrm{ev}_B, \mathrm{coev}_B)$ is a left dual of B. We can then define the *left dual* $f^* : B^* \to A^*$ as the unique morphism making the following diagram commute:

$$\begin{array}{ccccc} & B^* & \overset{f^*}{\dashrightarrow} & A^* & \\ \swarrow^{\rho_{B^*}^{-1}} & & & & \nwarrow^{\lambda_{A^*}} \\ B^* \otimes \mathbb{1} & & & & \mathbb{1} \otimes A^* \\ \downarrow{\scriptstyle \mathrm{id}_{B^*} \otimes \mathrm{coev}_A} & & & & \uparrow{\scriptstyle \mathrm{ev}_B \otimes \mathrm{id}_{A^*}} \\ B^* \otimes (A \otimes A^*) & \xrightarrow{\alpha^{-1}_{(B^*, A, A^*)}} & (B^* \otimes A) \otimes A^* & \xrightarrow{(\mathrm{id}_{B^*} \otimes f) \otimes \mathrm{id}_{A^*}} & (B^* \otimes B) \otimes A^* \end{array} \quad . \tag{3.23}$$

Graphically this is once again denoted by rotating the string diagrams upside down, i.e.

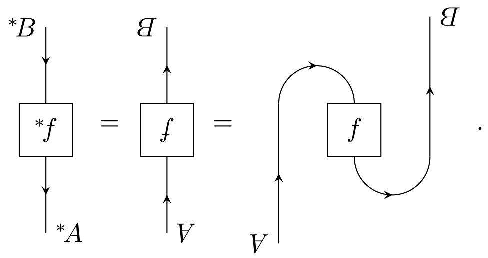


(3.24)

2. Suppose that $({}^*A, \mathrm{ev}_A, \mathrm{coev}_A)$ is a right dual of A, and that $({}^*B, \mathrm{ev}_B, \mathrm{coev}_B)$ is a right dual of B. We can then define the *right dual* ${}^*f : {}^*B \to {}^*A$ as the unique morphism making the following diagram commute:

$$\begin{array}{ccccc} & {}^*B & \overset{{}^*f}{\dashrightarrow} & {}^*A & \\ \swarrow^{\lambda_{{}^*B}^{-1}} & & & & \nwarrow^{\rho_{{}^*A}} \\ \mathbb{1} \otimes {}^*B & & & & {}^*A \otimes \mathbb{1} \\ \downarrow{\scriptstyle \mathrm{coev}_A \otimes \mathrm{id}_{{}^*B}} & & & & \uparrow{\scriptstyle \mathrm{id}_{{}^*A} \otimes \mathrm{ev}_{{}^*B}} \\ ({}^*A \otimes A) \otimes {}^*B & \xrightarrow{\alpha_{({}^*A, A, {}^*B)}} & {}^*A \otimes (A \otimes {}^*B) & \xrightarrow{\mathrm{id}_{{}^*A} \otimes (f \otimes \mathrm{id}_{{}^*B})} & {}^*A \otimes (B \otimes {}^*B) \end{array} \quad . \tag{3.25}$$

Graphically we can write this as



(3.26)

Example 29. Let $\mathcal{C}$ be a monoidal category with monoidal unit $\mathbb{1}$. $\mathbb{1}$ is its own (left and right) dual. Set $\mathrm{ev}_{\mathbb{1}} := \lambda_{\mathbb{1}}$ and $\mathrm{coev}_{\mathbb{1}} := \rho_{\mathbb{1}}^{-1}$, then the zigzag diagrams (3.14) and (3.15) commute due to the triangle identity (3.5).

Example 30 (Duals of modules). Let us consider the monoidal category of modules over some commutative ring R, ${}_R\mathbf{Mod}$. Suppose that the R-module M has some left dual $(M^*, \mathrm{ev}_M, \mathrm{coev}_M)$. We then know that $(\mathrm{id}_M \otimes \mathrm{ev}_M) \circ (\mathrm{coev}_M \otimes \mathrm{id}_M) = \mathrm{id}_M$.

We will first show that M has to be finitely generated, the proof for this statement is based on this answer on StackExchange ([Sta19]). Note that the coevaluation morphism $\mathrm{coev}_M : R \to M \otimes_R M^*$ ends up in a submodule of $M \otimes_R M^*$ that is generated by a single element $\sum_{i=1}^k m_i \otimes n_i$ for some $k \in \mathbb{N}$ and $m_i \in M, n_i \in M^*$. Let $\overline{M}, \overline{M^*}$ be the submodules of M, M^* generated by $\{m_1, \dots, m_k\}, \{n_1, \dots, n_k\}$. For the zigzag identity (3.14), we then find

$$M \longrightarrow R \otimes_R M \xrightarrow{\mathrm{coev}_M \otimes \mathrm{id}_M} \overline{M} \otimes_R \overline{M^*} \otimes_R M \xrightarrow{\mathrm{id}_{\overline{M}} \otimes \mathrm{ev}_M} \overline{M} \otimes_R R \longrightarrow \overline{M}\ , \tag{3.27}$$

so we see that we end up in a finitely generated submodule of M. However, this should be the identity morphism, so $M = \overline{M}$ is finitely generated. Similarly, we show that $M^* = \overline{M^*}$ is finitely generated.

Second, we will show that a finitely generated R-module that has a (left) dual M should also be projective. Suppose once again that $\mathrm{coev}_M(1_R) = \sum_{i=1}^k m_i \otimes n_i$, where we can now assume that $\{m_1, \dots, m_k\}, \{n_1, \dots, n_k\}$ are generating sets for M, M^*. Let $\{e_1, \dots, e_k\}$ be the standard generating set for R^k, we define

$$\pi : R^k \to M : \sum_{i=1}^k r_i e_i \mapsto \sum_{i=1}^k r_i m_i, \tag{3.28}$$

$$\sigma : M \to R^k : m \mapsto \sum_{i=1}^k \mathrm{ev}_M(n_i \otimes m) e_i. \tag{3.29}$$

We claim that $\pi \circ \sigma = \mathrm{id}_M$, indeed:

$$\begin{aligned}(\pi \circ \sigma)(m) &= \sum_{i=1}^k \mathrm{ev}_M(n_i \otimes m) m_i \\ &= \sum_{i=1}^k m_i \otimes \mathrm{ev}_M(n_i \otimes m) \\ &= (\mathrm{id}_M \otimes \mathrm{ev}_M) \left(\sum_{i=1}^k m_i \otimes n_i \otimes m \right) \\ &= ((\mathrm{id}_M \otimes \mathrm{ev}_M) \circ (\mathrm{coev}_M \otimes \mathrm{id}_M))(1_R \otimes m) \\ &= m\end{aligned} \tag{3.30}$$

due to the zigzag identity (3.14). We thus have a split epimorphism $R^k \to M$, which implies that M is a direct summand of R^k through Proposition 2.3.4. This shows that M is projective.

So, we know that modules that have duals must be finitely generated and projective. We will now show that every such module does indeed have a dual. For any R-module M, we define

$$M^* := \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, R) = \{R\text{-linear maps } M \to R\}. \tag{3.31}$$

There is an obvious candidate for an evaluation morphism (on any R-module, not just projective finitely generated ones) with this candidate dual module; ev_M is the morphism induced by the universal property of the tensor product on

$$\begin{array}{ccc} M^* \times M & \xrightarrow{\otimes} & M^* \otimes_R M \\ & \searrow^{\mathrm{eval}} & \downarrow{\scriptstyle \mathrm{ev}_M} \\ & & R \end{array}\quad , \text{ where } \mathrm{eval}(f, m) = f(m). \tag{3.32}$$

We know that M will have to be projective and finitely generated for M^* to be an actual dual, so we will now assume that M is projective and finitely generated.

As M is projective and finitely generated, we know that M is a direct summand of a finitely generated free R-module[4], i.e. there exists an R-module N and $k \in \mathbb{N}$ such that $M \oplus N \cong R^k$. Let $\{e_1, \ldots, e_k\}$ be the standard generating set for R^n.

For $i = 1, \ldots, n$, we have $e_i = m_i + n_i$ for some $m_i \in M$ and $n_i \in N$. It is then easy to see that $\{m_1, \ldots, m_k\}$ is a generating set for M. We can then define a generating set for M^* too by letting δ_i be the restriction of the map $R^k \to R : e_i \mapsto 1_R$ and $e_j \mapsto 0_R$ if $j \neq i$. For $m \in M$, we have $m = \sum_{i=1}^k \delta_i(m) e_i$, and thus also $m = \sum_{i=1}^k \delta_i(m) m_i$ (because $M \cap N = 0$ in R^k).

We can then define the coevaluation morphism as

$$\mathrm{coev}_M : R \to M \otimes_R M^* : r \mapsto r \sum_{i=1}^k m_i \otimes \delta_i. \tag{3.33}$$

Proving the zigzag identities is now easy, for example:

$$((\mathrm{id}_M \otimes \mathrm{ev}_M) \circ (\mathrm{coev}_M \otimes \mathrm{id}_M))(m) = (\mathrm{id}_M \otimes \mathrm{ev}_M) \left(\sum_{i=1}^k m_i \otimes \delta_i \otimes m \right) = \sum_{i=1}^k \delta_i(m) m_i = m. \tag{3.34}$$

It is clear that this dual (3.31) is both a left dual and a right dual by swapping around the tensor products. We will see later that this is because the category is equipped with a *symmetric braiding* (the swap map $x \otimes y \mapsto y \otimes x$).

Remark 3.4.2. Note that if M is a projective finitely generated module over a commutative ring R, then its dual M^* is also projective and finitely generated.

The projectivity of M^* follows from the general identity $(A \oplus B)^* \cong A^* \oplus B^*$, which holds in any abelian monoidal category where the monoidal product is bilinear on morphisms.

This identity implies in particular that $(R^k)^* \cong R^k$, since $R^* \cong R$ in the category of R-modules. Therefore, because projective finitely generated modules are direct summands of finite free modules, their duals are likewise direct summands of finite free modules, and hence projective and finitely generated.

Example 31 (Finite-dimensional vector spaces). Applying the above examples to fields, we see that the monoidal category of finite-dimensional vector spaces over any field is equipped with duals.

Example 32 (Finite-dimensional representations). Let G be a group and let $\mathbb{K}$ be a field. The category of finite-dimensional $\mathbb{K}$-linear G-representations $\mathbf{FinRep}_{\mathbb{K}}(G)$ has duals by setting $\rho^* : G \to \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V^*, V^*) : g \mapsto \rho(g^{-1})^*$ with evaluation morphism $\mathrm{ev}_{(V,\rho)} := \mathrm{ev}_V$ and coevaluation morphism $\mathrm{coev}_{(V,\rho)} := \mathrm{coev}_V$. It is easy to check the zigzag identities through $\rho(g^{-1})^*(\varphi)(\rho(g)(v)) = (\varphi \circ \rho(g^{-1}) \circ \rho(g))(v) = \varphi(v)$.

We will generalise this in § 6.4.2.

[4]The proof once again relies on Proposition 2.3.4. As M is finitely generated, there exists some epimorphism $R^k \to M$, and as M is projective this epimorphism splits. By Proposition 2.3.4, we then see that $R^k \cong M \oplus N$ for some N.

3.4.2 Properties of duals

As with any interesting categorical definition, duals are unique up to unique isomorphism.

Theorem 3.4.3 ([EGNO15, Proposition 2.10.5]). *Let $\mathcal{C}$ be a monoidal category. If an object $A \in \mathrm{Ob}(\mathcal{C})$ has a left (resp. right) dual, then it is unique up to unique isomorphism mapping the evaluation and coevaluation maps to evaluation and coevalutation maps.*

Proof. Let $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$ and $(\overline{A^*}, \overline{\mathrm{ev}_A}, \overline{\mathrm{coev}_A})$ be two left duals of A.

We will start by proving that, if an isomorphism $\mu : A^* \to \overline{A^*}$ exists and preserves the evaluation and coevaluation maps, this isomorphism is unique. The proof will give us a hint of how to define μ. The requirement that μ preserves the evaluation and coevaluation maps means that the following diagrams commute.

$$\begin{array}{ccc} A^* \otimes A & \xrightarrow{\mathrm{ev}_A} & \mathbb{1} \\ {\scriptstyle \mu\otimes\mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{1}}} \\ \overline{A^*} \otimes A & \xrightarrow{\overline{\mathrm{ev}_A}} & \mathbb{1} \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\mathrm{coev}_A} & A \otimes A^* \\ {\scriptstyle \mathrm{id}_{\mathbb{1}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A\otimes\mu} \\ \mathbb{1} & \xrightarrow{\overline{\mathrm{coev}_A}} & A \otimes \overline{A^*} \end{array} \tag{3.35}$$

We then find, using (3.35), functoriality of the associator (3.1), functoriality of $\otimes$, and functoriality of the left and right unitors (3.2) and (3.3), that

$$\begin{aligned} (\mathrm{ev}_A \otimes \mathrm{id}_{\overline{A^*}}) \circ \alpha^{-1}_{(A^*,A,\overline{A^*})} \circ (\mathrm{id}_{A^*} \otimes \overline{\mathrm{coev}_A}) &= (\mathrm{ev}_A \otimes \mathrm{id}_{\overline{A^*}}) \circ \alpha^{-1}_{(A^*,A,\overline{A^*})} \circ (\mathrm{id}_{A^*} \otimes ((\mathrm{id}_A \otimes \mu) \circ \mathrm{coev}_A)) \\ &= (\mathrm{ev}_A \otimes \mathrm{id}_{\overline{A^*}}) \circ (\mathrm{id}_{A^* \otimes A} \otimes \mu) \circ \alpha^{-1}_{(A^*,A,A^*)} \circ (\mathrm{id}_{A^*} \otimes \mathrm{coev}_A) \\ &= (\mathrm{id}_{\mathbb{1}} \otimes \mu) \circ ((\mathrm{ev}_A \otimes \mathrm{id}_{A^*}) \circ \alpha^{-1}_{(A^*,A,A^*)} \circ (\mathrm{id}_{A^*} \otimes \mathrm{coev}_A)) \\ &= (\mathrm{id}_{\mathbb{1}} \otimes \mu) \circ \lambda^{-1}_{A^*} \circ \rho_{A^*} \\ &= \lambda^{-1}_{\overline{A^*}} \circ \mu \circ \rho_{A^*} \end{aligned} \quad . \tag{3.36}$$

This means that μ is indeed uniquely defined, if it exists.

This suggests the following definition for $\mu : A^* \to \overline{A^*}$

$$A^* \xrightarrow{\rho^{-1}_{A^*}} A^* \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{A^*} \otimes \overline{\mathrm{coev}_A}} A^* \otimes (A \otimes \overline{A^*}) \xrightarrow{\alpha^{-1}_{(A^*,A,\overline{A^*})}} (A^* \otimes A) \otimes \overline{A^*} \xrightarrow{\mathrm{ev}_A \otimes \mathrm{id}_{\overline{A^*}}} \mathbb{1} \otimes \overline{A^*} \xrightarrow{\lambda_{\overline{A^*}}} \overline{A^*} \ . \tag{3.37}$$

Similarly, one defines a morphism $\nu : \overline{A^*} \to A^*$ through

$$\overline{A^*} \xrightarrow{\rho^{-1}_{\overline{A^*}}} \overline{A^*} \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{\overline{A^*}} \otimes \mathrm{coev}_A} \overline{A^*} \otimes (A \otimes A^*) \xrightarrow{\alpha^{-1}_{(\overline{A^*},A,A^*)}} (\overline{A^*} \otimes A) \otimes A^* \xrightarrow{\overline{\mathrm{ev}_A} \otimes \mathrm{id}_{A^*}} \mathbb{1} \otimes A^* \xrightarrow{\lambda_{A^*}} A^* \ . \tag{3.38}$$

Using (3.14) and the coherence conditions, it is not hard (but tedious) to find that μ satisfies (3.35).

We will now prove that μ and ν are inverses.

Let us consider the following diagram

$$\begin{array}{ccccc} A^* & \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} & A^* \otimes A \otimes A^* & \overset{\mathrm{id}}{\searrow} & \\ {\scriptstyle \mathrm{id}\otimes\mathrm{coev}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{coev}\otimes\mathrm{id}} & & \\ A^* \otimes A \otimes \overline{A^*} & \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} & A^* \otimes A \otimes \overline{A^*} \otimes A \otimes A^* & \xrightarrow{\mathrm{id}\otimes\mathrm{ev}\otimes\mathrm{id}} & A^* \otimes A \otimes A^* \\ {\scriptstyle \mathrm{ev}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} \\ \overline{A^*} & \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} & \overline{A^*} \otimes A \otimes A^* & \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} & A^* \end{array} \ , \tag{3.39}$$

where the associator, unitors and other specific morphisms are suppressed. We can then see that the three small rectangles commute. For example, it is clear that functoriality of $\otimes$ implies that the top left rectangle commutes

$$\begin{array}{ccc} (A^* \otimes \mathbb{1}) \otimes \mathbb{1} & \xrightarrow{(\mathrm{id}_{A^*} \otimes \mathrm{id}_{\mathbb{1}}) \otimes \mathrm{coev}_A} & (A^* \otimes \mathbb{1}) \otimes (A \otimes A^*) \\ {\scriptstyle (\mathrm{id}_{A^*} \otimes \overline{\mathrm{coev}_A}) \otimes \mathrm{id}_{\mathbb{1}}} \big\downarrow & & \big\downarrow {\scriptstyle (\mathrm{id}_{A^*} \otimes \overline{\mathrm{coev}_A}) \otimes \mathrm{id}_{A \otimes A^*}} \\ (A^* \otimes (A \otimes \overline{A^*})) \otimes \mathbb{1} & \xrightarrow{(\mathrm{id}_{A^*} \otimes \mathrm{id}_{A \otimes \overline{A^*}}) \otimes \mathrm{coev}_A} & (A^* \otimes (A \otimes \overline{A^*})) \otimes (A \otimes A^*) \end{array} . \tag{3.40}$$

Similarly, one shows that the other two rectangles commute.

The small triangle in the top right corner commutes due to the above definition for left duals and the evaluation, coevaluation morphisms: we have $\mathrm{id} \otimes \mathrm{ev} \circ \mathrm{coev} \otimes \mathrm{id} = \mathrm{id}$, and therefore $\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id} \circ \mathrm{id} \otimes \mathrm{coev} \otimes \mathrm{id} = \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} = \mathrm{id}$.

We can then see that the path following the left and lower boundary is the same as the one following the top and right boundary. That first path is equal to $\nu \circ \mu$, and that second path is the identity. We thus find that $\nu \circ \mu = \mathrm{id}$.

With a similar diagram, one finds $\mu \circ \nu = \mathrm{id}$. We conclude that $A^* \cong \overline{A^*}$. ■

Theorem 3.4.4 ([EGNO15, Remark 2.10.3]). *Let $\mathcal{C}$ be a monoidal category, and let $A \in \mathrm{Ob}(\mathcal{C})$. If A has a left (resp. right) dual $(A^*, \mathrm{ev}_A^{\mathrm{left}}, \mathrm{coev}_A^{\mathrm{left}})$ (resp. $({}^*A, \mathrm{ev}_A^{\mathrm{right}}, \mathrm{coev}_A^{\mathrm{right}})$), then this dual has a right (resp. left) dual $(A, \mathrm{ev}_A^{\mathrm{left}}, \mathrm{coev}_A^{\mathrm{left}})$ (resp. $(A, \mathrm{ev}_A^{\mathrm{right}}, \mathrm{coev}_A^{\mathrm{right}})$), i.e.*

$${}^*(A^*) \cong A \text{ and } ({}^*A)^* \cong A. \tag{3.41}$$

Proof. Commutativity of (3.19) follows from commutativity of (3.15). Alternatively, it is immediately clear from the graphical calculus. ■

Interestingly, duals provide us with adjoint pairs which show that the monoidal product preserves limits or colimits.

Proposition 3.4.5. *Let $\mathcal{C}$ be a monoidal category. For any object $A \in \mathrm{Ob}(\mathcal{C})$, define the functors*

$$A \otimes - : \mathcal{C} \to \mathcal{C} : X \mapsto A \otimes X \text{ and } f \mapsto \mathrm{id}_A \otimes f, \tag{3.42}$$
$$- \otimes A : \mathcal{C} \to \mathcal{C} : X \mapsto X \otimes A \text{ and } f \mapsto f \otimes \mathrm{id}_A. \tag{3.43}$$

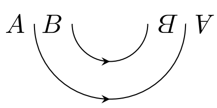
If A has a left (resp. right) dual, then we have the adjoint pairs $(A^* \otimes -, A \otimes -)$ and $(- \otimes A, - \otimes A^*)$ (resp. $(A \otimes -, {}^*A \otimes -)$ and $(- \otimes {}^*A, - \otimes A)$).

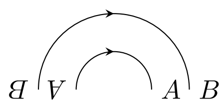
Proof. For the adjunction $(A^* \otimes -, A \otimes -)$, we set $\eta_X := (\mathrm{coev}_A \otimes \mathrm{id}_X) \circ \lambda_X^{-1}$ and $\varepsilon_X : \lambda_X \circ (\mathrm{ev}_A \otimes \mathrm{id}_X)$. The adjunction equations on these natural transformations then boil down to the zigzag equations. ■

Duals also interact well with the monoidal product.

Lemma 3.4.6. *Let $\mathcal{C}$ be a monoidal category, and let $A, B \in \mathrm{Ob}(\mathcal{C})$. Suppose that A, B have left (resp. right) duals $(A^*, \mathrm{ev}_A, \mathrm{coev}_A), (B^*, \mathrm{ev}_B, \mathrm{coev}_B)$ (resp. $({}^*A, \mathrm{ev}_A, \mathrm{coev}_A), ({}^*B, \mathrm{ev}_B, \mathrm{coev}_B)$). $A \otimes B$ then has a left (resp. right) dual $(B^* \otimes A^*, \mathrm{ev}_A \circ (\lambda_A \otimes \mathrm{id}_{A^*}) \circ (\mathrm{id}_A \otimes \mathrm{ev}_B \otimes \mathrm{id}_{A^*}), (\mathrm{id}_A \otimes \mathrm{coev}_B \otimes \mathrm{id}_{A^*}) \circ (\lambda_A^{-1} \otimes \mathrm{id}_{A^*}) \circ \mathrm{coev}_A)$ (resp. $({}^*B \otimes {}^*A, \mathrm{ev}_A \circ (\lambda_{{}^*A} \otimes \mathrm{id}_A) \circ (\mathrm{id}_{{}^*A} \otimes \mathrm{ev}_B \otimes \mathrm{id}_A), (\mathrm{id}_{{}^*A} \otimes \mathrm{coev}_B \otimes \mathrm{id}_A) \circ (\lambda_{{}^*A}^{-1} \otimes \mathrm{id}_A) \circ \mathrm{coev}_A)$).*

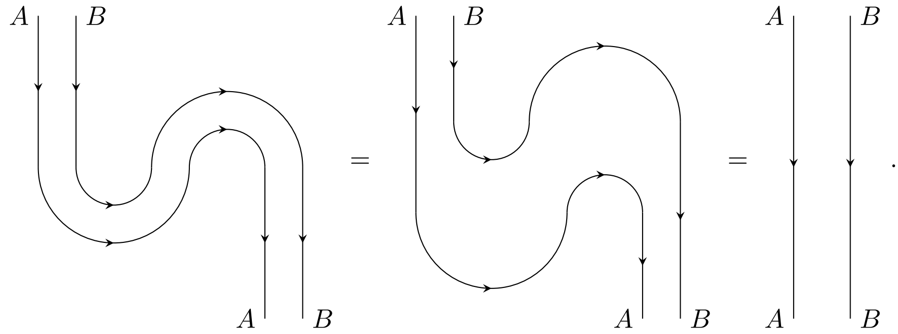
Proof. We can prove this through graphical calculus. The evaluation and coevaluation morphisms we have defined are

$\mathrm{ev}_{A \otimes B} =$ 

and $\mathrm{coev}_{A \otimes B} =$ 

. (3.44)

To prove the zigzag equations we then use the zigzag equations for A and B, for example



(3.45)

3.4.3 Rigid categories and dualisation functors

Categories in which all objects admit duals (e.g. categories of finitely generated projective modules over commutative rings) are quite special.

Definition 3.4.7 (Rigid monoidal categories, [EGNO15, Definition 2.10.11]). A monoidal category is called *rigid* (resp. *left, right rigid*) if every object has both a left and a right (resp. a left, right) dual.

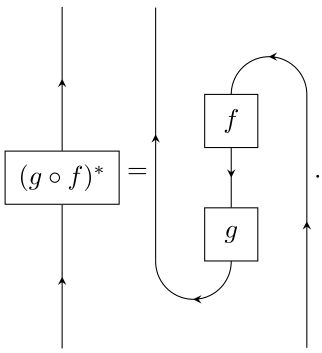
On such categories, dualisation defines a functor.

Proposition 3.4.8 (Dualisation functor). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal category. Any choice of left duals for any object defines a* left *(resp.* right*)* dualisation functor *(this is a contravariant functor)*

$$-^* : \mathcal{C}^{\text{dual}} \to \mathcal{C}^{\text{op}} : A \mapsto A^* \text{ and } f \mapsto f^* \ (\text{resp. } {}^*- : \mathcal{C}^{\text{dual}} \to \mathcal{C} : A \mapsto {}^*A \text{ and } f \mapsto {}^*f). \tag{3.46}$$

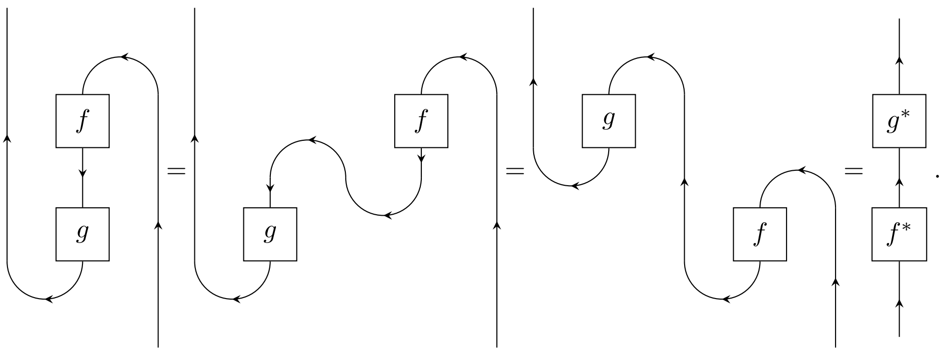
Moreover, this functor is monoidal.

Proof. Note that $\text{id}_{A^*} = \text{id}_A{}^*$ is immediate from the definition of duals of morphisms (3.24) and the zigzag identity (3.18). All that is left for us to prove, is that $(g \circ f)^* = f^* \circ g^*$. By definition, we have



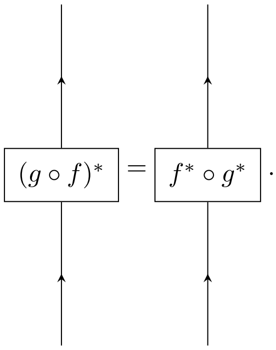
(3.47)

Applying the zigzag identity (3.18), we then find



(3.48)

We conclude that



(3.49)

Finally, Lemma 3.4.6 implies that this functor is monoidal. ■

Corollary 3.4.9. *Let $\mathcal{C}$ be a rigid monoidal category. Any pair of a left and a right dualisation functor forms a pair of monoidal quasi-inverses (which means that there exist monoidal natural isomorphisms between their compositions and the identity functor).*

Proof. Let F be any left dualisation functor, and let G be any right dualisation functor. Theorem 3.4.3 and Theorem 3.4.4 show that we find unique isomorphisms

$$\eta_A : A \to (G \circ F)(A) \text{ and } \varepsilon_A : (F \circ G)(A) \to A \text{ for any } A \in \mathrm{Ob}\,(\mathcal{C})\,. \tag{3.50}$$

Naturality follows from the definition of the unique isomorphism in Theorem 3.4.3, and the fact that this natural isomorphism is monoidal follows from Lemma 3.4.6. ■

Remark 3.4.10. Due to the above corollary, we obtain adjoint pairs $(-^*, {}^*-)$ (where $-^*$ is interpreted as a functor $\mathcal{C}^{\mathrm{dual}} \to \mathcal{C}$, and ${}^*-$ is interpreted as a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{dual}}$) and $({}^*-, -^*)$ (where ${}^*-$ is interpreted as a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{dual}}$, and $-^*$ is interpreted as a functor $\mathcal{C}^{\mathrm{dual}} \to \mathcal{C}$). As a consequence, $-^*$ and ${}^*-$ map limits (resp. colimits) in $\mathcal{C}$ to colimits (resp. limits) in $\mathcal{C}$.

In particular, they are exact when $\mathcal{C}$ is abelian and the monoidal product is bilinear on morphisms. This not necessarily true when the category is only left or right rigid! On left rigid multiring categories, for example, this is a non-trivial statement we will prove later (in Proposition 4.3.1).

Similarly, we obtain the adjoint pairs $((-^*)^*, {}^*({}^*-))$ and $({}^*({}^*-), (-^*)^*)$. This implies that these functors preserve limits and colimits.

3.5 Traces, pivotal, and spherical structures

3.5.1 Categorical traces

Duals in monoidal categories allow us to introduce a notion of traces in categories. Such a trace is sometimes called a *categorical* or *quantum trace*.

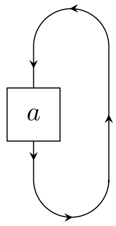
Definition 3.5.1 (Traces in monoidal categories, [EGNO15, Definition 4.7.1]). Let $\mathcal{C}$ be a monoidal category, and let $A \in \mathrm{Ob}\,(\mathcal{C})$.

1. Suppose that A has a left dual A^*, suppose that A^* has a left dual A^{**}, and let $a \in \mathrm{Hom}_{\mathcal{C}}(A, A^{**})$. The *left trace* of a is defined as the composition

$$\mathrm{tr}_A^{\mathrm{left}}(a) = \mathrm{tr}^{\mathrm{left}}(a) : \; \mathbb{1} \xrightarrow{\mathrm{coev}_A^{\mathrm{left}}} A \otimes A^* \xrightarrow{a \otimes \mathrm{id}_{A^*}} A^{**} \otimes A^* \xrightarrow{\mathrm{ev}_{A^*}^{\mathrm{left}}} \mathbb{1}\;. \tag{3.51}$$

Using the graphical calculus, this is

$$\mathrm{tr}_A^{\mathrm{left}}(a) = \mathrm{tr}^{\mathrm{left}}(a) = \quad (3.52)$$



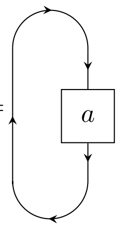
which can easily be remembered by the fact that the morphism should be placed on the left side for the left trace.

2. Suppose that A has a right dual ${}^*A$, suppose that ${}^*A$ has a right dual ${}^{**}A$, and let $a \in \mathrm{Hom}_{\mathcal{C}}(A, {}^{**}A)$. The *right trace* of a is defined as the composition

$$\mathrm{tr}_A^{\mathrm{right}}(a) = \mathrm{tr}^{\mathrm{right}}(a) : \mathbb{1} \xrightarrow{\mathrm{coev}_A^{\mathrm{right}}} {}^*A \otimes A \xrightarrow{\mathrm{id}_{{}^*A} \otimes a} {}^*A \otimes {}^{**}A \xrightarrow{\mathrm{ev}_{{}^*A}^{\mathrm{right}}} \mathbb{1} \ . \quad (3.53)$$

Using the graphical calculus, this is

$$\mathrm{tr}_A^{\mathrm{right}}(a) = \mathrm{tr}^{\mathrm{right}}(a) = \quad (3.54)$$



which can easily be remembered by the fact that the morphism should be placed on the right side for the right trace.

Remark 3.5.2. Note that traces are morphisms $\mathbb{1} \to \mathbb{1}$. We can thus only interpret traces as scalars in some ring or field when $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a ring or a field. We will show, in § 4.1, that this happens if and only if the category is enriched over some commutative ring R such that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \cong R$.

Example 33 (Traces on projective finitely generated modules). Let R be a commutative ring, and let M be a projective finitely generated R-module with generating set $\{m_1, \ldots, m_k\}$. From Example 30, we know that M has a dual $M^* = \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, R)$ (which is once again projective and finitely generated, and is thus dualisable). In this case, however, we find that $M^{**} \cong M$ because left and right duals are the same thing (we can then use Theorem 3.4.4).

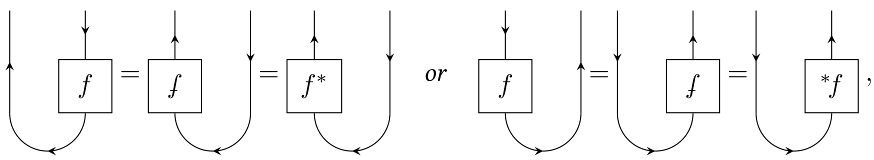
Let $a : M \to M$ be an R-linear map, and suppose that $a_{ij} = \delta_j(a(m_i))$. The trace of a in the above sense is then the morphism

$$\mathrm{tr}_M(a) : R \to R : r \mapsto r \sum_{i=1}^{k} \delta_i(a(m_i)) = r \sum_{i=1}^{k} a_{ii}. \quad (3.55)$$

We want to show that categorical traces share some of the important properties with the standard trace, as one would expect from the above Example 33.

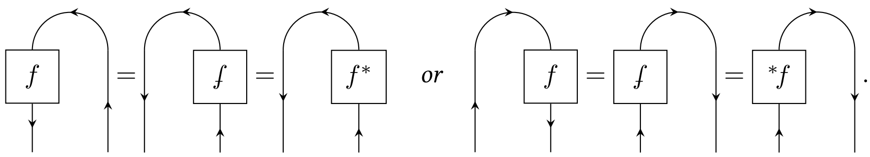
Before we can do this, we will prove a little lemma that shows that morphisms can be rotated along curves in the graphical calculus.

Lemma 3.5.3. *Let $\mathcal{C}$ be a monoidal category, let $A, B \in \mathrm{Ob}(\mathcal{C})$, and let $f : A \to B$ in $\mathcal{C}$. If left duals $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$ and $(B^*, \mathrm{ev}_B, \mathrm{coev}_B)$ or right duals $({}^*A, \mathrm{ev}_A, \mathrm{coev}_A)$ and $({}^*B, \mathrm{ev}_B, \mathrm{coev}_B)$ of A and B exist, then f can be rotated along curves in the following way*



(3.56)

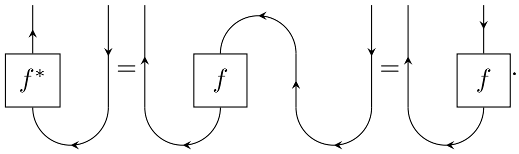
and



(3.57)

Note that clockwise rotations result in left duals, and anticlockwise rotations in right duals.

Proof. We will only prove the first equality of (3.56) using graphical calculus. From (3.18) and (3.24), we obtain



(3.58)

■

Proposition 3.5.4 (Properties of the traces, [EGNO15, Proposition 4.7.3]). *Let $\mathcal{C}$ be a category that is left (resp. right) rigid, let $A, B \in \mathrm{Ob}(\mathcal{C})$, and let $a \in \mathrm{Hom}_{\mathcal{C}}(A, A^{**})$, $b \in \mathrm{Hom}_{\mathcal{C}}(B, B^{**})$, $c \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ (resp. $a \in \mathrm{Hom}_{\mathcal{C}}(A, {}^{**}A)$, $b \in \mathrm{Hom}_{\mathcal{C}}(B, {}^{**}B)$). The following properties hold*

1. $\mathrm{tr}^{\mathrm{left}}_A(a) = \mathrm{tr}^{\mathrm{right}}_A(a^*)$ *(resp.* $\mathrm{tr}^{\mathrm{right}}_A(a) = \mathrm{tr}^{\mathrm{left}}_A({}^*a)$*),*
2. $\mathrm{tr}^{\mathrm{left}}_{A\otimes B}(a \otimes b) = \mathrm{tr}^{\mathrm{left}}_A(a) \otimes \mathrm{tr}^{\mathrm{left}}_B(b)$ *(resp.* $\mathrm{tr}^{\mathrm{right}}_{A\otimes B}(a \otimes b) = \mathrm{tr}^{\mathrm{right}}_A(a) \otimes \mathrm{tr}^{\mathrm{right}}_B(b)$*),*
3. $\mathrm{tr}^{\mathrm{left}}_A(a \circ c) = \mathrm{tr}^{\mathrm{left}}_A(c^{**} \circ a)$ *(resp.* $\mathrm{tr}^{\mathrm{right}}_A(a \circ c) = \mathrm{tr}^{\mathrm{right}}_A({}^{**}c \circ a)$*).*
4. *if, in addition, $\mathcal{C}$ is additive and the monoidal product is bilinear on morphisms, then* $\mathrm{tr}^{\mathrm{left}}_{A\oplus B}(a \oplus b) = \mathrm{tr}^{\mathrm{left}}_A(a) + \mathrm{tr}^{\mathrm{left}}_B(b)$ *(resp.* $\mathrm{tr}^{\mathrm{right}}_{A\oplus B}(a \oplus b) = \mathrm{tr}^{\mathrm{right}}_A(a) + \mathrm{tr}^{\mathrm{right}}_B(b)$*).*

Proof. Lemma 3.5.3 implies (1) and (3). The second property (2) is immediately obvious from the graphical calculus and Lemma 3.4.6. For the last property (4), note that $\mathrm{tr}^{\mathrm{left}}_A : \mathrm{Hom}_{\mathcal{C}}(A, A^{**}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is linear due to the composition and monoidal product being bilinear. Now, $\mathrm{tr}^{\mathrm{left}}_{A\oplus B}(a \oplus b) = \mathrm{tr}^{\mathrm{left}}_{A\oplus B}(i_A{}^{**} \circ a \circ p_A + i_B{}^{**} \circ b \circ p_B) = \mathrm{tr}^{\mathrm{left}}_A(p_A{}^{**} \circ i_A{}^{**} \circ a) + \mathrm{tr}^{\mathrm{left}}_B(p_B{}^{**} \circ i_B{}^{**} \circ b) = \mathrm{tr}^{\mathrm{left}}_A(a) + \mathrm{tr}^{\mathrm{left}}_B(b)$ (and a similar proof holds for right traces). ■

3.5.2 Pivotal categories

The categorical traces we have defined above are not entirely satisfactory for two reasons:

1. we have two different notions of a categorical trace on rigid monoidal categories: a left and a right trace,
2. we take traces of morphisms to double duals, not of endomorphisms as we would expect.

It turns out that the general definition needs to be stated this way, and that these issues are only resolved in specific types of rigid monoidal categories.

First we will introduce categories which solve the second issue.

Definition 3.5.5 (Pivotal structures on monoidal categories, [EGNO15, Definition 4.7.8]). Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is a monoidal natural isomorphism $\alpha : \mathrm{id}_{\mathcal{C}} \to -^{**}$, i.e. a collection of isomorphisms $\alpha_A : A \to A^{**}$ for all $A \in \mathrm{Ob}(\mathcal{C})$ such that $\alpha_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, $\alpha_{A\otimes B} = \alpha_A \otimes \alpha_B$ for all $A, B \in \mathrm{Ob}(\mathcal{C})$, and such that the following diagram commutes for all $f : A \to B$ in $\mathcal{C}$

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & A^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ B & \xrightarrow{\alpha_B} & B^{**} \end{array} . \tag{3.59}$$

A monoidal category equipped with a pivotal structure is called a *pivotal category*.

Remark 3.5.6. The pivotal structure in the above definition could be called a left pivotal structure, and one could introduce a right pivotal structure similarly. However, Theorem 3.4.4 ensures that, on rigid categories, a left pivotal structure induces a right pivotal structure and vice versa. Indeed, $\alpha_{^{**}A} : {}^{**}A \to ({}^{**}A)^{**} \cong ({}^{*}A)^{*} \cong A$, taking the inverse yields a right pivotal structure $\beta_A = \alpha^{-1}_{^{**}A}$.

Remark 3.5.7. Equivalently, a pivotal structure on $\mathcal{C}$ is a monoidal natural isomorphism $\zeta : {}^{*}- \to -^{*}$. This follows from Proposition 3.4.4 by setting $\zeta_A := {}^{*}\alpha_A : {}^{*}A \to A^{*}$.

In pivotal categories, we have a stronger version of Proposition 3.5.4 (3).

Lemma 3.5.8. *Let $\mathcal{C}$ be a pivotal category with pivotal structure α. Let $A, B \in \mathrm{Ob}(\mathcal{C})$, let $f : A \to B$, and let $g : B \to A$. We have*

$$\mathrm{tr}^{\mathrm{left}}_A(\alpha_A \circ g \circ f) = \mathrm{tr}^{\mathrm{left}}(\alpha_B \circ f \circ g) \text{ and } \mathrm{tr}^{\mathrm{right}}_A(g \circ f \circ \alpha_A^{-1}) = \mathrm{tr}^{\mathrm{right}}_B(f \circ g \circ \alpha_B^{-1}). \tag{3.60}$$

Proof. This follows from Lemma 3.5.3. ■

Lemma 3.5.9 ([EGNO15, Exercise 4.7.9]). *Let $\mathcal{C}$ be a rigid monoidal category, let α be a pivotal structure on $\mathcal{C}$, and let $A \in \mathrm{Ob}(\mathcal{C})$. Then*

$$\alpha_{A^*} = \left(\alpha_A{}^{*}\right)^{-1}. \tag{3.61}$$

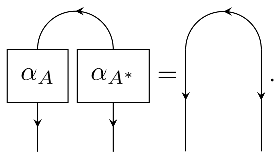
From this, we conclude

$$\alpha_{A^{**}} = \left((\alpha_{A^*})^{*}\right)^{-1} = \left(\left((\alpha_A{}^{*})^{-1}\right)^{*}\right)^{-1} = \alpha_A{}^{**}. \tag{3.62}$$

Proof. The commutativity of

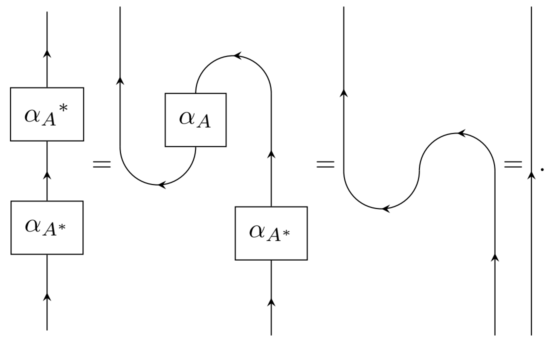
$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\alpha_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}} & \mathbb{1} \\ {\scriptstyle \mathrm{coev}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{coev}_A{}^{**}} \\ A \otimes A^{*} & \xrightarrow{\alpha_{A\otimes A^*} = \alpha_A \otimes \alpha_{A^*}} & (A \otimes A^{*})^{**} \end{array} \tag{3.63}$$

implies that



(3.64)

As a consequence, we obtain



(3.65)

■

Traces also allow us to introduce an intrinsic notion of *dimensions of objects.*

Definition 3.5.10 (Dimensions of objects, [EGNO15, Definition 4.7.11]). Let $\mathcal{C}$ be a pivotal category with pivotal structure α (β is defined as in Remark 3.5.6), and let $A \in \mathrm{Ob}(\mathcal{C})$. The *left* (resp. *right*) *dimension* of A is defined as

$$\dim^{\mathrm{left}}_{\alpha}(A) := \mathrm{tr}^{\mathrm{left}}_A(\alpha_A) \text{ (resp. } \dim^{\mathrm{right}}_{\alpha}(A) := \mathrm{tr}^{\mathrm{right}}_A(\beta_A)). \tag{3.66}$$

Remark 3.5.11. We would like to note that this intrinsic notion of dimension does not necessarily coincide with the notion of dimension we are used to. For example, if $\mathbb{K}$ is a field of characteristic $p > 0$, then $\mathbb{K}^p$ is a vector space of dimension p in the usual sense, and of dimension 0 in the categorical sense (as the categorical dimension is a scalar in $\mathbb{K}$, or rather a morphism $\mathbb{K} \to \mathbb{K}$, and not an integer).

Example 34. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a pivotal category. We have

$$\dim^{\text{left}}(\mathbb{1}) = \dim^{\text{right}}(\mathbb{1}) = \text{id}_{\mathbb{1}}. \tag{3.67}$$

Indeed, Example 29 shows that $\text{ev}_{\mathbb{1}} = \lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ and $\text{coev}_{\mathbb{1}} = \text{ev}_{\mathbb{1}}^{-1}$. As $\alpha_{\mathbb{1}} = \text{id}_{\mathbb{1}}$ for the pivotal structure, we then find $\text{tr}^{\text{left}}(\alpha_{\mathbb{1}}) = \lambda_{\mathbb{1}} \circ \lambda_{\mathbb{1}}^{-1} = \text{id}_{\mathbb{1}}$.

More generally, for $f : \mathbb{1} \to \mathbb{1}$, we have

$$\text{tr}_{\mathbb{1}}^{\text{left}}(f) = \rho_{\mathbb{1}} \circ (f \otimes \text{id}_{\mathbb{1}}) \circ \rho_{\mathbb{1}}^{-1} = f \text{ and } \text{tr}_{\mathbb{1}}^{\text{right}}(f) = \lambda_{\mathbb{1}} \circ (\text{id}_{\mathbb{1}} \otimes f) \circ \lambda_{\mathbb{1}}^{-1} = f. \tag{3.68}$$

Remark 3.5.12. Let $\mathcal{C}$ be a pivotal category with pivotal structure α. When are the left and right dimensions the same, i.e. when does $\text{tr}^{\text{left}}(\alpha_A) = \text{tr}^{\text{right}}(\beta_A)$ hold for an object $A \in \text{Ob}(\mathcal{C})$?

Similarly to Lemma 3.5.9, we obtain $\beta_{*A} = ({}^*\beta_A)^{-1}$, and thus $\alpha_{***A} = ({}^*\alpha_{**A})^{-1}$. Using this, ${}^*\!-^* = \text{id}_{\mathcal{C}}$, Proposition 3.5.4, and Lemma 3.5.9, we obtain

$$\begin{aligned} \text{tr}^{\text{right}}(\beta_A) &= \text{tr}^{\text{right}}\left(\alpha_{**A}^{-1}\right) \\ &= \text{tr}^{\text{right}}\left(\left(({}^*\alpha_{**A})^{-1}\right)^*\right) \\ &= \text{tr}^{\text{right}}\left((\alpha_{***A})^*\right) \\ &= \text{tr}^{\text{left}}\left(\alpha_{***A}\right) \\ &= \text{tr}^{\text{left}}\left((\alpha_{***A})^{**}\right) \\ &= \text{tr}^{\text{left}}\left(\alpha_{*A}\right) \\ &= \text{tr}^{\text{left}}\left((\alpha_{*A})^{**}\right) \\ &= \text{tr}^{\text{left}}\left(\alpha_{A^*}\right). \end{aligned} \tag{3.69}$$

We thus find

$$\dim_\alpha^{\text{left}}(A) = \dim_\alpha^{\text{right}}(A) \text{ if and only if } \dim_\alpha^{\text{left}}(A) = \dim_\alpha^{\text{left}}(A^*). \tag{3.70}$$

3.5.3 Spherical categories

Finally, we introduce categories in which the left and right categorical traces coincide.

Definition 3.5.13 (Spherical structures on monoidal categories, [BW99, Definition 2.5]). Let $\mathcal{C}$ be a rigid monoidal category, and let α be a pivotal structure on $\mathcal{C}$. α is called *spherical* if, for any object $A \in \text{Ob}(\mathcal{C})$, and any endomorphism $f : A \to A$, we have

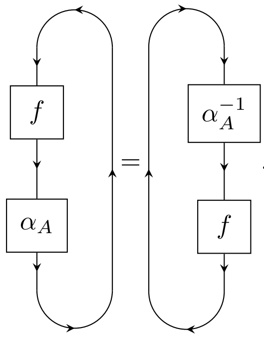
$$\text{tr}^{\text{left}}(\alpha_A \circ f) = \text{tr}^{\text{right}}(f \circ \alpha_A^{-1}). \tag{3.71}$$

A rigid monoidal category $\mathcal{C}$ equipped with a spherical structure is called a *spherical category*. For any endomorphism $f : A \to A$ in $\mathcal{C}$, we define the *trace* as

$$\text{tr}(f) = \text{tr}_A(f) := \text{tr}^{\text{left}}(\alpha_A \circ f) = \text{tr}^{\text{right}}(f \circ \alpha_A^{-1}), \tag{3.72}$$

i.e.

$$\mathrm{tr}(f) =$$



(3.73)

Remark 3.5.14. In monoidal categories that are also abelian and where the monoidal and abelian structures are compatible, commonly referred to as *tensor categories* (see Chapter 4), one can alternatively define spherical categories through Remark 3.5.12. Specifically, this means requiring that

$$\dim_\alpha^{\mathrm{left}}(A) = \dim_\alpha^{\mathrm{left}}(A^*) \text{ for all objects } A \in \mathrm{Ob}\,(\mathcal{C})\,. \tag{3.74}$$

This formulation of spherical tensor categories appears in [EGNO15, Definition 4.7.14]. Furthermore, [EGNO15, Theorem 4.7.15] shows that this definition is equivalent to the one we have introduced. One direction of the equivalence follows directly from Proposition 3.5.4 and Remark 3.5.12, using the identities

$$\mathrm{tr}^{\mathrm{right}}(\alpha_A^{-1}) = \mathrm{tr}^{\mathrm{left}}({}^*\alpha_A^{-1}) = \mathrm{tr}^{\mathrm{left}}(\alpha_A^{-1*}) = \mathrm{tr}^{\mathrm{left}}(\alpha_{A^*}). \tag{3.75}$$

3.6 Braided monoidal categories

3.6.1 Braidings

In Example 30 we discussed how the duals of modules are both left and right modules, and that this is because of the existence of a "swap map" on the tensor product of modules. In this section we will generalise this to the setting of braidings.

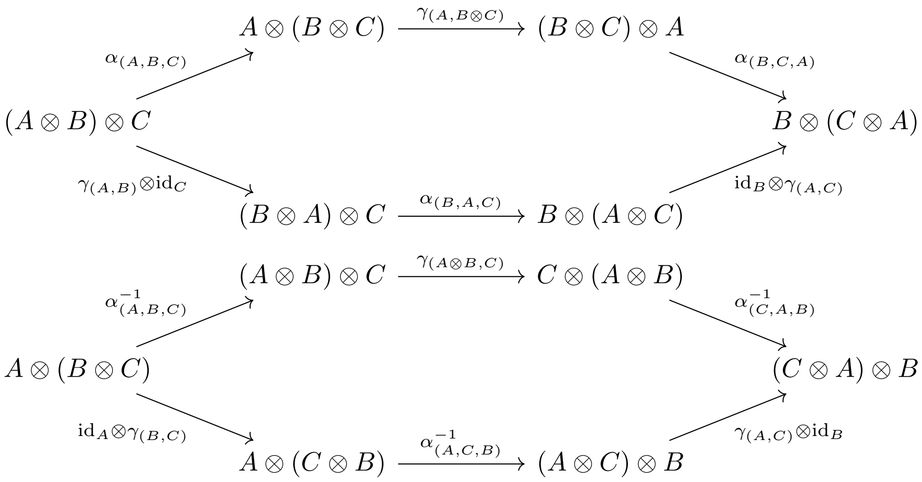
Braidings will be particularly important for us because they enable us to address the commutativity of algebras within categories (as will be discussed in Chapter 6).

Remark 3.6.1. From the perspective of categorification, braided monoidal categories arise naturally as categorified analogues of commutative monoids.

Definition 3.6.2 (Braidings on monoidal categories, [Lan78, § XI.1] and [EGNO15, Definition 8.1.1]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. A *braiding* (or *commutativity constraint*, which is a term often used in older papers such as [DM82; Del90; Del02]) on this monoidal category is a natural isomorphism $\gamma : \otimes \to \otimes^{\mathrm{op}}$, i.e. a class of isomorphisms $\gamma_{(A,B)} : A \otimes B \to B \otimes A$ such that the following diagram commutes for all $f : A \to B$ and $g : X \to Y$ in $\mathcal{C}$

$$\begin{array}{ccc} A \otimes X & \xrightarrow{\gamma_{(A,X)}} & X \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ B \otimes Y & \xrightarrow{\gamma_{(B,Y)}} & Y \otimes B \end{array}\,, \tag{3.76}$$

such that, in addition, the following diagrams commute for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$

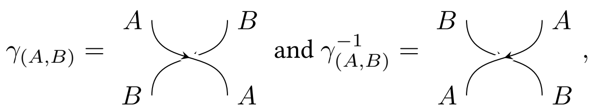


(3.77)

These identities are called the *hexagon identities.*

The tuple $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ is called a *braided monoidal category.*

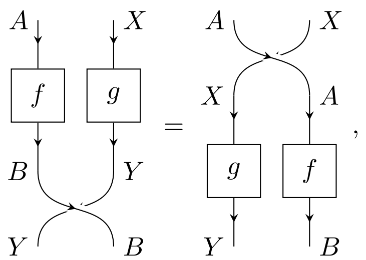
Graphically, we denote braidings and their inverses by letting the lines cross



(3.78)

note that which edge crosses over the other one determines whether we are working with the braiding or its inverse.

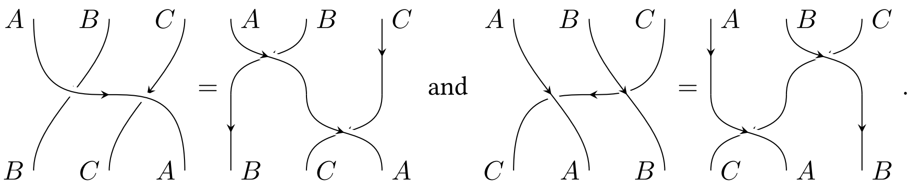
Naturality of the braiding then becomes



(3.79)

which shows that we can "slide" morphisms through braidings.

The hexagon identities graphically become



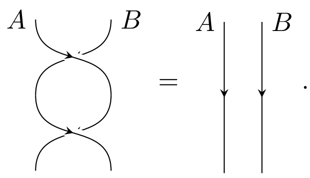
(3.80)

Example 35 (The reverse of a braided monoidal category). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a braided monoidal category. We can define a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma^{\mathrm{rev}})$, called the *reverse braided monoidal category,* through the reverse monoidal braiding $\gamma^{\mathrm{rev}}_{(A,B)} := \gamma^{-1}_{(B,A)}$. Note that both γ and γ^{rev} are monoidal braidings on the same monoidal category.

Definition 3.6.3 (Symmetric categories, [Lan78, § XI.1] and [EGNO15, Definition 8.1.12]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a braided monoidal category. It is called a *symmetric category*, and γ is called a *symmetric braiding*, if

$$\gamma_{(B,A)} \circ \gamma_{(A,B)} = \mathrm{id}_{A \otimes B} \text{ for all } A, B \in \mathrm{Ob}(\mathcal{C}), \text{ i.e. if } \gamma^{-1}_{(B,A)} = \gamma_{(A,B)}. \tag{3.81}$$

Graphically, this is



(3.82)

Note that the reverse braiding on a monoidal category is the original braiding if and only if that original braiding is symmetric.

Example 36 (Swap map on modules). Let R be a commutative ring. The monoidal category of R-modules ${}_R\mathbf{Mod}$ can be equipped with a braiding, called the *swap map*, defined by the universal property of the tensor product

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes N \\ & \searrow^{s} & \downarrow{\scriptstyle \mathrm{swap}_{(M,N)}} \\ & & N \otimes M \end{array}, \quad \text{where } s : M \times N \to N \otimes M : (m,n) \mapsto n \otimes m. \tag{3.83}$$

It is easy to see that $\mathrm{swap}^{-1}_{(M,N)} = \mathrm{swap}_{(N,M)}$, which implies that ${}_R\mathbf{Mod}$ equipped with the swap map is symmetric.

Example 37 (Signed swap map on representations, [EGNO15, Example 9.9.1 (3)]). Let G be a group, let $\mathbb{K}$ be a field, and let $\mathbf{FinRep}_{\mathbb{K}}(G)$ be the monoidal category of $\mathbb{K}$-linear finite-dimensional G-representations. Suppose that there is some central element $z \in G$ such that $z^2 = 1_G$. For any representation, we find a basis such that for all m in this basis $\rho(z)(m) = (-1)^{\delta_m} m$ with $\delta_m \in \{0, 1\}$. We define the *signed swap map* through the universal property of the tensor product

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes N \\ & \searrow^{s} & \downarrow{\scriptstyle \mathrm{signedswap}_{((M,\rho),(N,\sigma))}} \\ & & N \otimes M \end{array}, \quad \text{where } s : M \times N \to N \otimes M : (m,n) \mapsto (-1)^{\delta_m \delta_n} n \otimes m. \tag{3.84}$$

We can check that signedswap is a morphism in $\mathbf{FinRep}_{\mathbb{K}}(G)$, i.e. that

$$\mathrm{signedswap}_{((M,\rho),(N,\sigma))} \circ (\rho(g) \otimes \sigma(g)) = (\sigma(g) \otimes \rho(g)) \circ \mathrm{signedswap}_{((N,\sigma),(M,\rho))} \text{ for all } g \in G. \tag{3.85}$$

It is once again easy to see that $\mathrm{signedswap}^{-1}_{((M,\rho),(N,\sigma))} = \mathrm{signedswap}_{((N,\sigma),(M,\rho))}$.

$\mathbf{FinRep}_{\mathbb{K}}(G)$ equipped with this braiding is denoted by $\mathbf{FinRep}_{\mathbb{K}}(G, z)$.

Example 38 (Super-vector spaces). Setting $G = \mathbb{Z}/2\mathbb{Z}$ and z the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ in the above Example 37, we retrieve the braided monoidal category of super-vector spaces $\mathbf{FinsVect}_{\mathbb{K}}$.

The structure-preserving functors for braided monoidal categories are called *braided monoidal functors*.

Definition 3.6.4 (Braided monoidal functors, [EGNO15, Definition 8.1.7]). Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha_{\mathcal{C}}, \lambda_{\mathcal{C}}, \rho_{\mathcal{C}}, \gamma_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}}, \gamma_{\mathcal{D}})$ be braided monoidal categories, and let (F, ζ, ε) be a monoidal functor between the underlying monoidal categories. This monoidal functor is called *braided* if the following diagram commutes for all $A, B \in \mathrm{Ob}(\mathcal{C})$

$$\begin{array}{ccc} F(A)\otimes_{\mathcal{D}} F(B) & \xrightarrow{(\gamma_{\mathcal{D}})_{(F(A),F(B))}} & F(B)\otimes_{\mathcal{D}} F(A) \\ {\scriptstyle \varepsilon_{(A,B)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon_{(B,A)}} \\ F(A\otimes_{\mathcal{C}} B) & \xrightarrow{F((\gamma_{\mathcal{C}})_{(A,B)})} & F(B\otimes_{\mathcal{C}} A) \end{array} . \tag{3.86}$$

Although we have not assumed braidings to be particularly well-behaved with regard to the monoidal structure (we did not mention unitors, ...), we will now show that braidings automatically interact well with the monoidal unit.

Lemma 3.6.5 ([Kas94, Proposition XIII.1.2] and [EGNO15, Exercise 8.1.6]). *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a braided monoidal category. For all $A \in \mathrm{Ob}(\mathcal{C})$, we have*

$$\lambda_A \circ \gamma_{(A,\mathbb{1})} = \rho_A \text{ and } \rho_A \circ \gamma_{(\mathbb{1},A)} = \lambda_A. \tag{3.87}$$

As a consequence, we find $\gamma_{(\mathbb{1},A)} = \gamma^{-1}_{(A,\mathbb{1})}$. In particular, $\gamma_{(\mathbb{1},\mathbb{1})} = \mathrm{id}_{\mathbb{1}}$.

Proof. Let $A, B \in \mathrm{Ob}(\mathcal{C})$. We claim that the following diagram is commutative

$$\begin{tikzcd}
(A\otimes\mathbb{1})\otimes B \arrow[r, "\alpha_{(A,\mathbb{1},B)}"] \arrow[dd, "\gamma_{(A,\mathbb{1})}\otimes\mathrm{id}_B"'] \arrow[dr, "\rho_A\otimes\mathrm{id}_B"'] & A\otimes(\mathbb{1}\otimes B) \arrow[r, "\gamma_{(A,\mathbb{1}\otimes B)}"] \arrow[d, "\mathrm{id}_A\otimes\lambda_B"] & (\mathbb{1}\otimes B)\otimes A \arrow[d, "\lambda_B\otimes\mathrm{id}_A"] \arrow[dr, "\alpha_{(\mathbb{1},B,A)}"] & \\
& A\otimes B \arrow[r, "\gamma_{(A,B)}"] & B\otimes A & \mathbb{1}\otimes(B\otimes A) \arrow[l, "\lambda_{B\otimes A}"'] \\
(\mathbb{1}\otimes A)\otimes B \arrow[r, "\alpha_{(\mathbb{1},A,B)}"] \arrow[ur, "\lambda_A\otimes\mathrm{id}_B"] & \mathbb{1}\otimes(A\otimes B) \arrow[r, "\mathrm{id}_{\mathbb{1}}\otimes\gamma_{(A,B)}"] \arrow[u, "\lambda_{A\otimes B}"] & \mathbb{1}\otimes(B\otimes A) \arrow[u, "\lambda_{B\otimes A}"] \arrow[ur, "\mathrm{id}_{\mathbb{1}\otimes(B\otimes A)}"'] &
\end{tikzcd} . \tag{3.88}$$

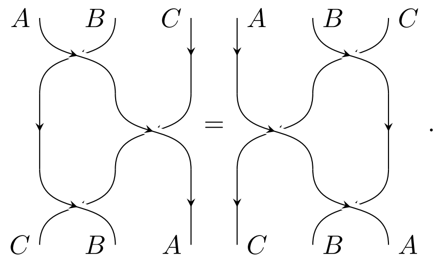
Indeed, the commutativity of every little square or triangle, except for the left one, follows from one of the axioms of monoidal categories or from the naturality of the braiding. The left triangle then automatically commutes too by composing with the isomorphism $\gamma_{(A,B)}$, using the commutativity of the other parts, and the hexagon identity (3.77).

Setting $B = \mathbb{1}$, we find

$$(\lambda_A \circ \gamma_{(A,\mathbb{1})}) \otimes \mathrm{id}_{\mathbb{1}} = \rho_A \otimes \mathrm{id}_{\mathbb{1}}. \tag{3.89}$$

This implies that $\lambda_A \circ \gamma_{(A,\mathbb{1})} = \rho_{\mathbb{1}} \circ (\rho_A \otimes \mathrm{id}_{\mathbb{1}}) \circ \rho_{\mathbb{1}}^{-1} = \rho_A$. ■

Theorem 3.6.6 (Yang-Baxter, [Kas94, Theorem XIII.1.3] and [EGNO15, Proposition 8.1.10]). *Let $\mathcal{C}$ be a monoidal category equipped with a braiding γ. For any $A, B, C \in \mathrm{Ob}(\mathcal{C})$, the following* Yang-Baxter equation *holds*



(3.90)

Proof. Through the first hexagon identity (3.77), we find

$$(\gamma_{(B,C)} \otimes \mathrm{id}_A) \circ (\mathrm{id}_B \otimes \gamma_{(A,C)}) \circ (\gamma_{(A,B)} \otimes \mathrm{id}_C) = (\gamma_{(B,C)} \otimes \mathrm{id}_A) \circ \gamma_{(A,B\otimes C)}, \tag{3.91}$$

and through the naturality of the braiding γ, we then find

$$(\gamma_{(B,C)} \otimes \mathrm{id}_A) \circ \gamma_{(A,B\otimes C)} = \gamma_{(A,B\otimes C)} \circ (\mathrm{id}_A \otimes \gamma_{(B,C)}). \tag{3.92}$$

Using the hexagon identity for the inverse braiding, we then find

$$\gamma_{(A,B\otimes C)} \circ (\mathrm{id}_A \otimes \gamma_{(B,C)}) = (\mathrm{id}_B \otimes \gamma_{(A,C)}) \circ (\gamma_{(A,B)} \otimes \mathrm{id}_C) \circ (\mathrm{id}_A \otimes \gamma_{(B,C)}) = (\mathrm{id}_B \otimes \gamma_{(A,C)}) \circ \gamma_{(A\otimes B,C)}. \tag{3.93}$$

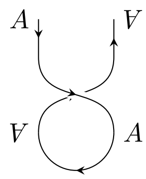
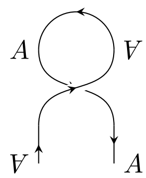
■

3.6.2 Braidings and duals

Braidings allows us to define right (resp. left) duals from left (resp. right) duals, which is what we already noted in Example 30.

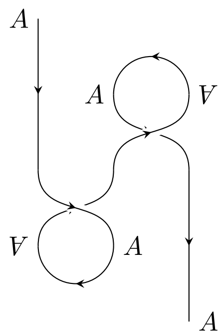
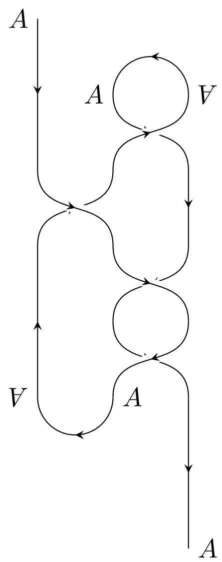
Proposition 3.6.7 ([Sel10, Lemma 3]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal category with a braiding γ. $\mathcal{C}$ is then also right (resp. left) rigid, hence rigid.*

Explicitly, if $A \in \mathrm{Ob}(\mathcal{C})$ has a left dual $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$, it is immediately also equipped with a right dual $(A^*, \overline{\mathrm{ev}_A}, \overline{\mathrm{coev}_A})$, where

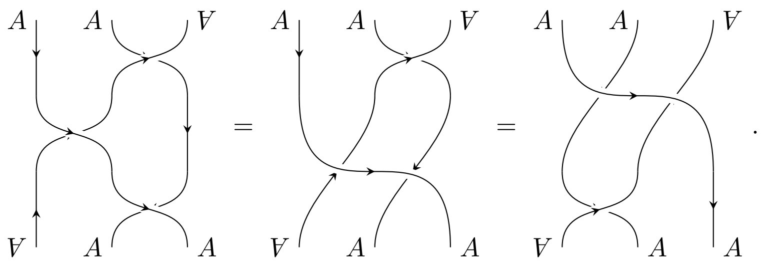
$\overline{\mathrm{ev}_A} =$  *and* $\overline{\mathrm{coev}_A} =$  . (3.94)

Proof. Suppose that A is an object with a left dual $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$, and let $(A^*, \overline{\mathrm{ev}_A}, \overline{\mathrm{coev}_A})$ be as defined above.

Writing out the left side of the first zigzag equation for right duals (3.22), and inserting $(\mathrm{id}_{A^*} \otimes \gamma^{-1}_{(A,A)}) \circ (\mathrm{id}_{A^*} \otimes \gamma_{(A,A)})$ in the middle, we obtain

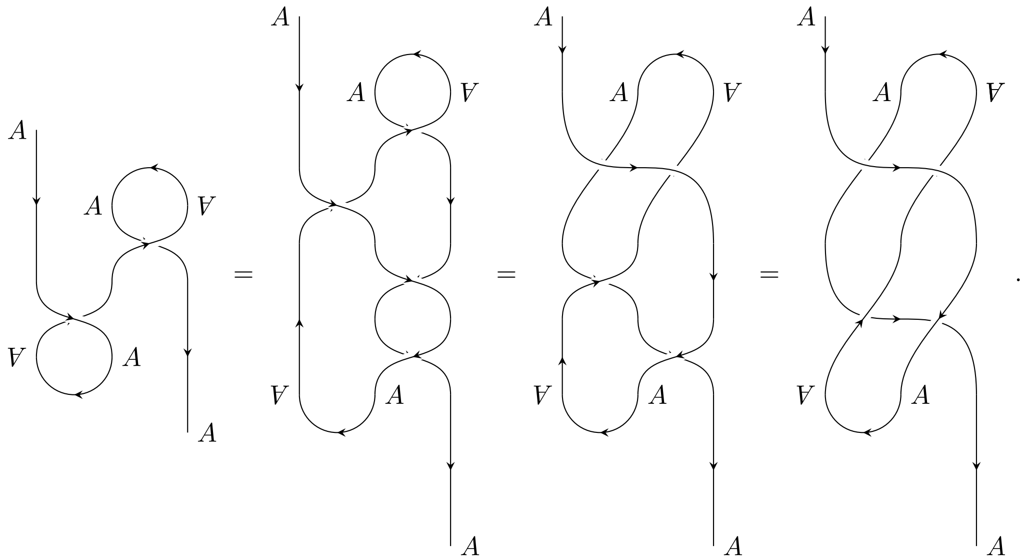
 =  . (3.95)

Through the hexagon identities (3.77) and the naturality of the braiding γ, we then see that (as in the proof of the Yang-Baxter equation, Theorem 3.6.6)



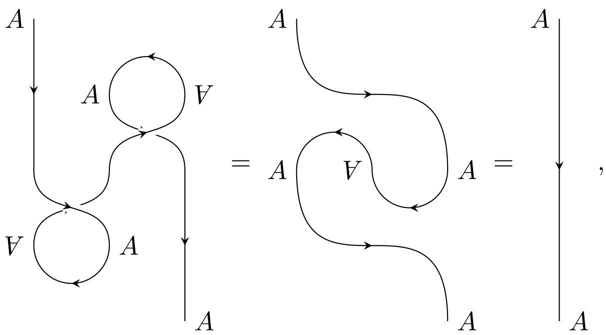
(3.96)

This implies that



(3.97)

Applying naturality of the braiding once again, we find



(3.98)

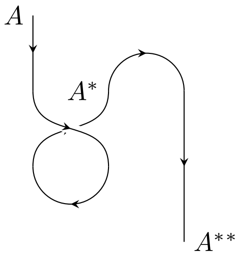
where we used the zigzag equation for the left dual (3.18), and the fact that the top braiding is $\gamma_{(A,\mathbb{1})}$ and the bottom one is $\gamma^{-1}_{(A,\mathbb{1})}$. ■

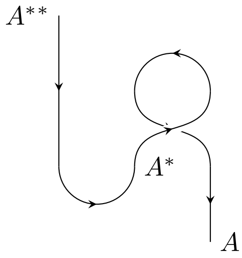
Corollary 3.6.8 ([Sel10, § 4.4.5] and [EGNO15, Proposition 8.10.6]). *Let $\mathcal{C}$ be a rigid monoidal category equipped with a braiding γ. There is a natural isomorphism*

$$\omega : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}{}^{**}, \tag{3.99}$$

called the Drinfeld morphism.

Explicitly, the natural isomorphism and its inverse are given by

$\omega_A =$ 

and $\omega_A^{-1} =$ 

. (3.100)

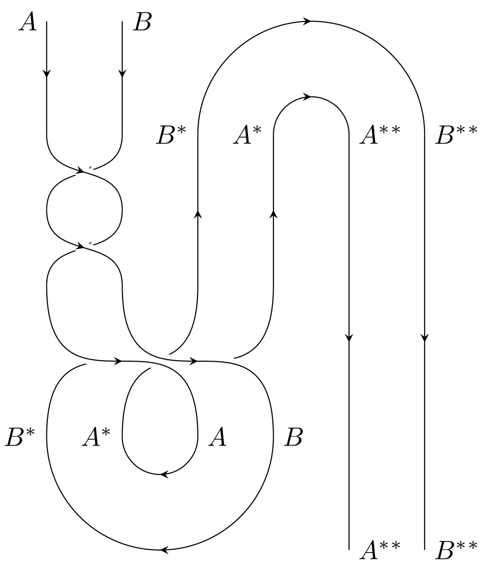
Proof. This follows from the proof of Proposition 3.6.7 directly, or equivalently from Proposition 3.4.3 and the statement of Proposition 3.6.7 by noting that both A and A^{**} are left duals of A^* due to Theorem 3.4.4. Naturality follows from Lemma 3.5.3 and naturality of the braiding (3.79). ■

Corollary 3.6.8 seems to indicate that there might be a pivotal structure on most rigid braided monoidal categories. However, it is not guaranteed that the natural isomorphism defined here is monoidal! In Theorem 3.6.11 below, we will show that a braided category is pivotal if and only if there is a natural isomorphism $\mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$ that is monoidal up to the braiding.

Lemma 3.6.9 ([EGNO15, Proposition 8.9.3]). *Let $\mathcal{C}$ be a rigid monoidal category with a braiding γ. The Drinfeld natural isomorphism $\omega : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}{}^{**}$, defined in Corollary 3.6.8, satisfies*

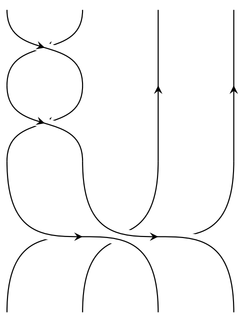
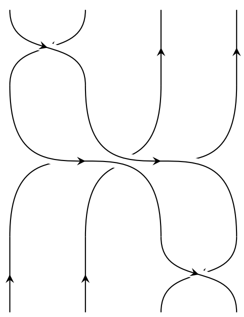
$$\omega_A \otimes \omega_B = \omega_{A\otimes B} \circ \gamma_{(B,A)} \circ \gamma_{(A,B)} \text{ for all } A, B \in \mathrm{Ob}(\mathcal{C})\,. \tag{3.101}$$

Proof. Graphically, we have

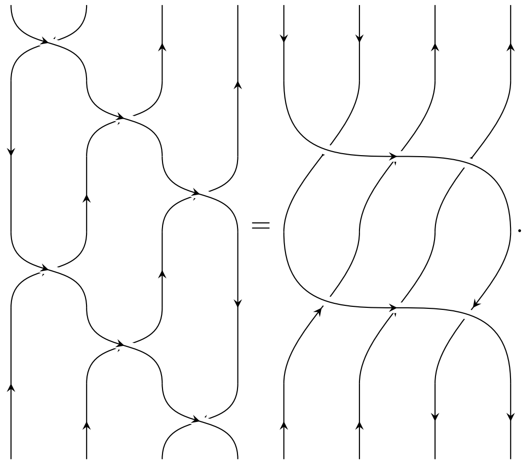
$\omega_{A\otimes B} \circ \gamma_{(B,A)} \circ \gamma_{(A,B)} =$ 

. (3.102)

We have


$=$

, (3.103)

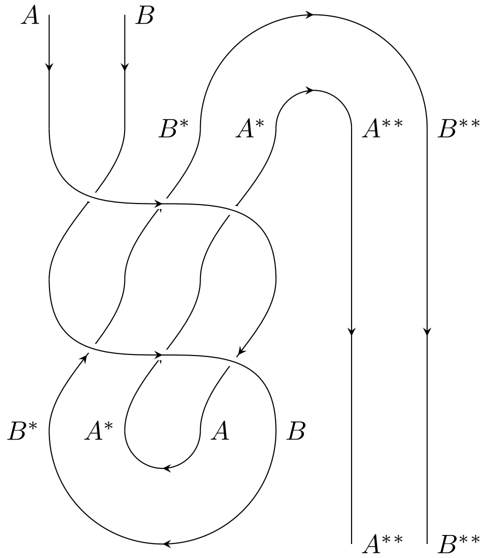
and splitting up the braiding $\gamma_{(A\otimes B, B^*\otimes A^*)}$ in this diagram gives



. (3.104)

This then implies that

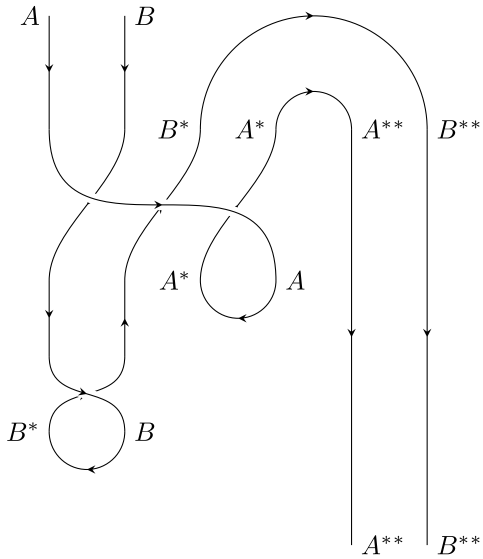
$$\omega_{A\otimes B} \circ \gamma_{(B,A)} \circ \gamma_{(A,B)} =$$



. (3.105)

Applying the naturality of the braiding (3.79) then gives

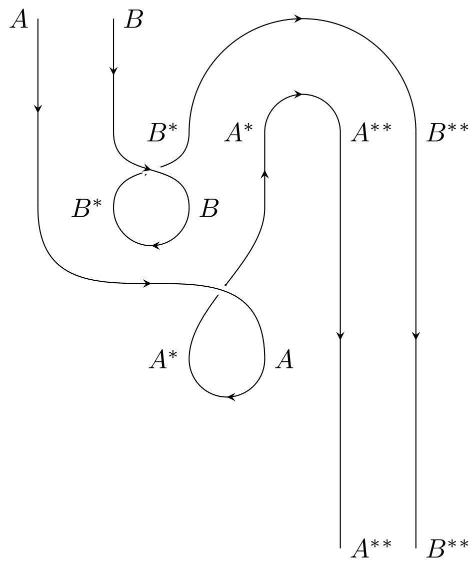
$$\omega_{A\otimes B} \circ \gamma_{(B,A)} \circ \gamma_{(A,B)} =$$



, (3.106)

and applying it again finally shows that

$$\omega_{A\otimes B} \circ \gamma_{(B,A)} \circ \gamma_{(A,B)} =$$



$$= \omega_A \otimes \omega_B . \qquad (3.107)$$

■

Definition 3.6.10 (Twists, [Sel10, Definition 17] and [EGNO15, Definition 8.10.1]). Let $\mathcal{C}$ be a monoidal category equipped with a braiding γ. A *twist* (or *balance*) is a natural isomorphism $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ such that

$$\theta_{A\otimes B} = \gamma_{(B,A)} \circ \gamma_{(A,B)} \circ (\theta_A \otimes \theta_B) \text{ for all } A, B \in \mathrm{Ob}(\mathcal{C}) . \qquad (3.108)$$

A braided monoidal category equipped with a twist is called a *balanced monoidal category.*

If, in addition, the category is rigid, and $\theta_A{}^* = \theta_{A^*}$ for all $A^* \in \mathrm{Ob}(\mathcal{C})$, then the twist is called a *ribbon*, and the category is called a *ribbon category.*

The following remarkable theorem shows that braidings that allow twists induce pivotal structures on the category.

Theorem 3.6.11 ([Sel10, Lemma 4] and [EGNO15, § 8.10]). *Let $\mathcal{C}$ be a rigid monoidal category equipped with a braiding γ. Any twist θ on this braided monoidal category induces a pivotal structure and every pivotal structure induces a twist.*

In particular; every rigid symmetric monoidal category is pivotal by choosing $\theta = \mathrm{id}$.

Proof. This follows easily from the above Lemma 3.6.9. Suppose that we are provided with a twist θ, then we can define the natural isomorphism

$$\alpha_A := \omega_A \circ \theta_A : A \to A^{**} \text{ for } A \in \mathrm{Ob}(\mathcal{C}) . \qquad (3.109)$$

For all $A, B \in \mathrm{Ob}(\mathcal{C})$, we then find

$$\alpha_A \otimes \alpha_B = (\omega_A \otimes \omega_B) \circ (\theta_A \otimes \theta_B) = \omega_{A\otimes B} \circ \gamma_{(B,A)} \circ \gamma_{(A,B)} \circ (\theta_A \otimes \theta_B) = \omega_{A\otimes B} \circ \theta_{A\otimes B} = \alpha_{A\otimes B} . \qquad (3.110)$$

Conversely, suppose that α is a pivotal structure on $\mathcal{C}$. We can then define

$$\theta_A := \omega_A^{-1} \circ \alpha_A \text{ for } A \in \mathrm{Ob}(\mathcal{C}) , \qquad (3.111)$$

and for all $A, B \in \mathrm{Ob}(\mathcal{C})$ we see that

$$\theta_{A\otimes B} = \omega_{A\otimes B}^{-1} \circ \alpha_{A\otimes B} = \gamma_{(B,A)} \circ \gamma_{(A,B)} \circ (\omega_A^{-1} \otimes \omega_B^{-1}) \circ (\alpha_A \otimes \alpha_B) = \gamma_{(B,A)} \circ \gamma_{(A,B)} \circ (\theta_A \otimes \theta_B) . \qquad (3.112)$$

■

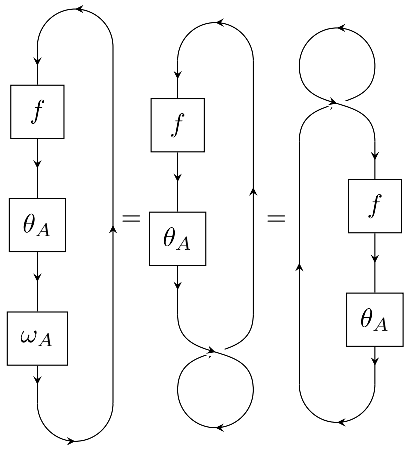
Corollary 3.6.12 ([Sel10, Lemma 5] and [EGNO15, Proposition 8.10.12]). *Let $\mathcal{C}$ be a rigid monoidal category with a braiding γ. If, in addition, $\mathcal{C}$ is equipped with a ribbon structure θ, then $\mathcal{C}$, equipped with the pivotal structure defined in Theorem 3.6.11, is spherical.*

In particular; every rigid symmetric monoidal category is spherical.

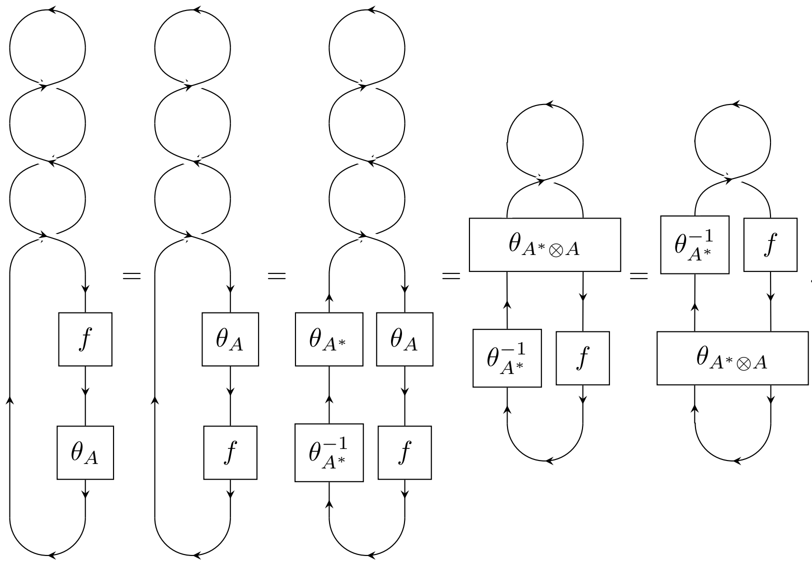
Proof. Let $\alpha = \omega \circ \theta$ be the pivotal structure defined in Theorem 3.6.11, and let $f : A \to A$ be any endomorphism in $\mathcal{C}$. As θ is ribbon, Lemma 3.5.9 implies that

$$\omega_A{}^* = (\alpha_A \circ \theta_A^{-1})^* = \theta_{A^*}^{-1} \circ \alpha_A{}^* = \theta_{A^*}^{-1} \circ \alpha_{A^*}^{-1}. \tag{3.113}$$

Applying the naturality of the braiding (3.79), we then find

$\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ f) =$  . (3.114)

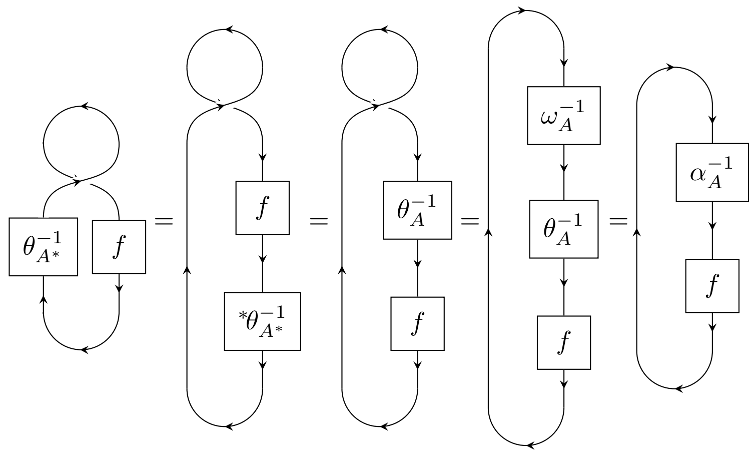
We want to obtain the right trace, so we should introduce the evaluation and coevaluation (3.94). Inserting $\gamma_{(A^*,A)} \circ \gamma_{(A^*,A)}^{-1}$, and using the naturality and twist property (3.108) of θ, we find

$\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ f) =$  . (3.115)

Using Lemma 3.6.5 and the twist property (3.108), we find $\theta_{\mathbb{1}\otimes\mathbb{1}} = \theta_{\mathbb{1}} \otimes \theta_{\mathbb{1}}$. Proposition 4.1.1 then shows that

$\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. As a consequence, we find (after applying naturality of θ once again)

$\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ f) =$



$= \mathrm{tr}^{\mathrm{right}}(f \circ \alpha_A^{-1})$. (3.116)

■

3.6.3 Action of the braid group on monoidal powers

Following [EGNO15, § 8.2], we will show that there is a natural action of the braid group on monoidal powers in braided monoidal categories.

Definition 3.6.13 (Braid groups). For any $n \in \mathbb{N}$, we define the *braid group on n strands* as the group with the presentation

$$B_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i < n-1, \text{ and } \sigma_m \sigma_n = \sigma_n \sigma_m \text{ for all } |m-n| > 1 \rangle. \tag{3.117}$$

Theorem 3.6.14 ([EGNO15, Remark 8.2.5]). *Let $\mathcal{C}$ be a braided monoidal category. For any object $A \in \mathrm{Ob}(\mathcal{C})$, and any $n \geq 1$, there is a natural action of the braid group B_n on $A^{\otimes n} = \otimes_{i=1}^{n} A$ through automorphisms, i.e. there is a group morphism*

$$B_n \longrightarrow \mathrm{Aut}_{\mathcal{C}}(A^{\otimes n}) \ . \tag{3.118}$$

If the braiding is symmetric, then this group morphism factors through the symmetric group S_n, i.e.

$$B_n \xrightarrow{\sigma_i^2 = 1} S_n \longrightarrow \mathrm{Aut}_{\mathcal{C}}(A^{\otimes n}) \ . \tag{3.119}$$

Proof. For $i = 1, \dots, n-1$, define $\sigma_i := \mathrm{id}_{A^{\otimes(i-1)}} \otimes \gamma_{(A,A)} \otimes \mathrm{id}_{A^{\otimes(n-i-1)}}$. The Yang-Baxter equation (Theorem 3.6.6) shows that

$$\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} \text{ for all } i < n-1, \tag{3.120}$$

and functoriality of the monoidal product also shows that

$$\sigma_m \circ \sigma_n = \sigma_n \circ \sigma_m \text{ for all } |m-n| > 1. \tag{3.121}$$

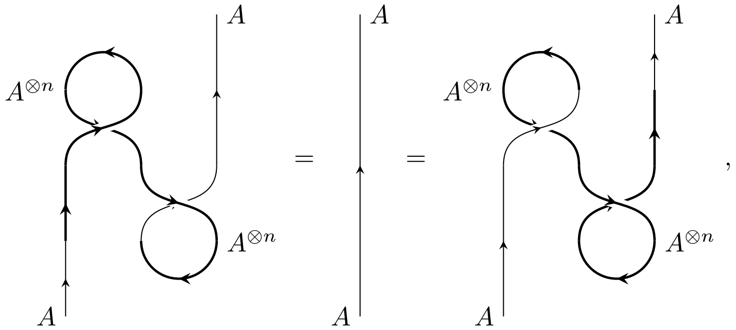
The final statement follows from the fact that the Coxeter presentation of the symmetric group is

$$S_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i < n-1, \sigma_m \sigma_n = \sigma_n \sigma_m \text{ for } |m-n| > 1, \sigma_i^2 = 1 \text{ for all } i \rangle. \tag{3.122}$$

■

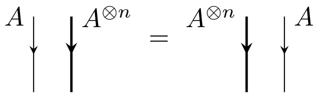
In the remainder of this section, we compute the trace of arbitrary elements of S_n acting on $A^{\otimes n}$.

Lemma 3.6.15. *Let $\mathcal{C}$ be a left rigid symmetric monoidal category, and let $A \in \mathrm{Ob}(\mathcal{C})$. For any $n \geq 1$, we have*



(3.123)

where thicker lines correspond to $A^{\otimes n}$ and

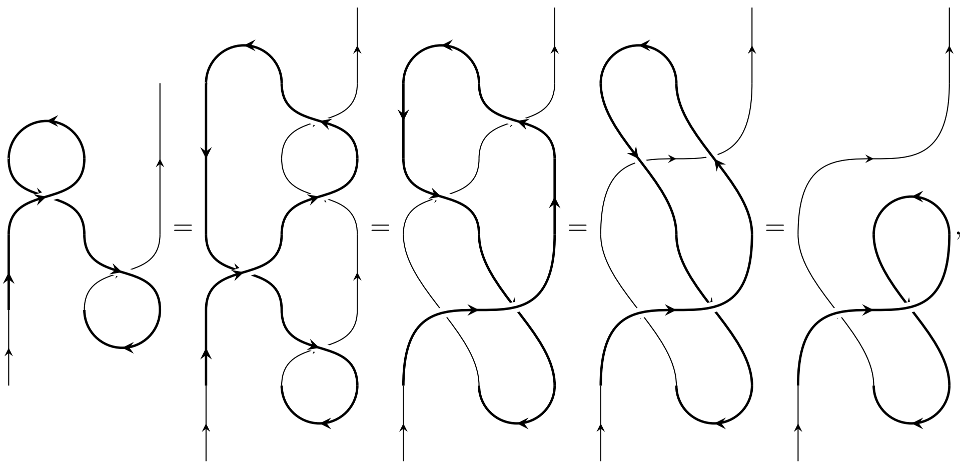


(3.124)

by regrouping (i.e. rebracketing of $A^{\otimes(n+1)}$).

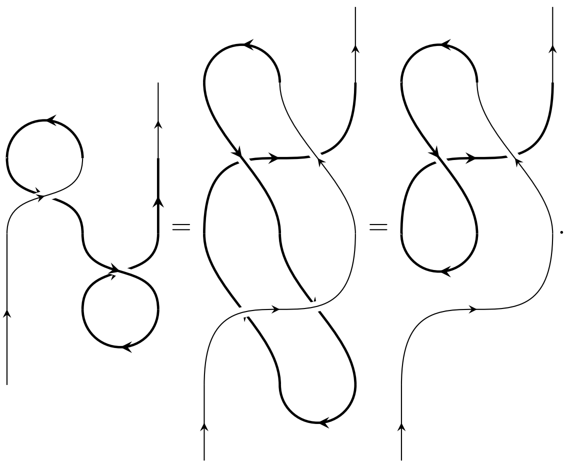
Proof. The base case $n = 1$ follows from Proposition 3.6.7.

Suppose now that $n > 1$. As in the proof of Proposition 3.6.7, we have



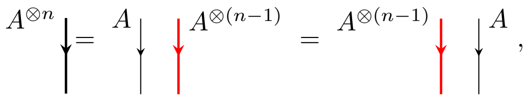
(3.125)

and similarly



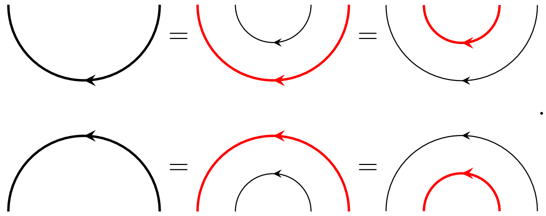
(3.126)

Decomposing



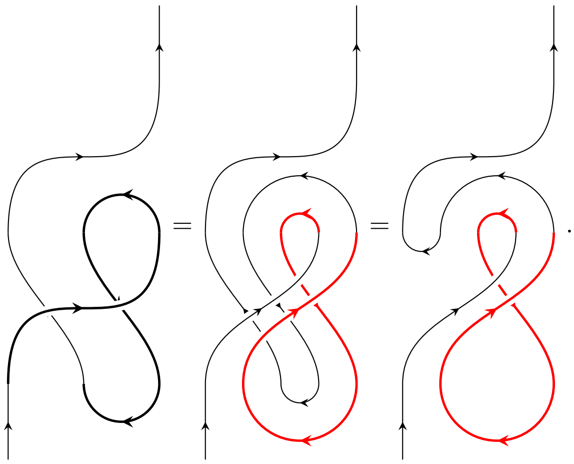
(3.127)

Lemma 3.4.6 shows that



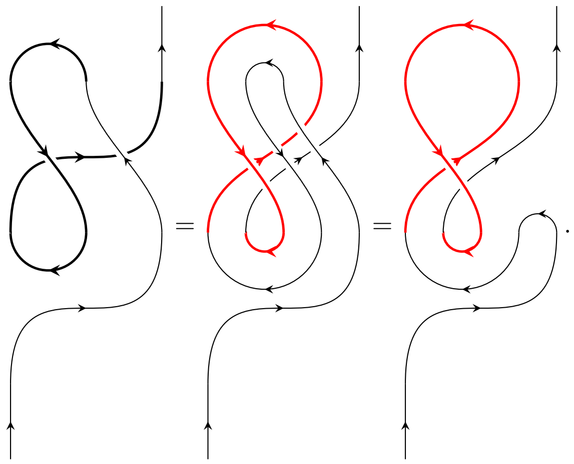
(3.128)

Inserting this into (3.125), and using the naturality of the braiding (3.79), we find



(3.129)

For (3.126), this gives



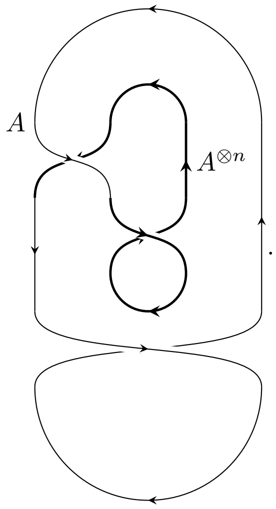
(3.130)

We then conclude the result by induction (using the fact that the braiding is symmetric). ■

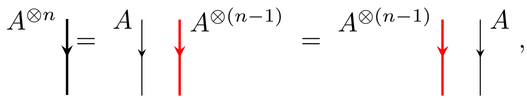
Proposition 3.6.16. *Let $\mathcal{C}$ be a rigid symmetric monoidal category, let $n \geq 1$, and let $\sigma := \gamma_{(A,A^{\otimes n})}$ (σ is thus a cyclic permutation acting on $A^{\otimes(n+1)}$). Provided with the canonical spherical structure on $\mathcal{C}$ induced by the twist $\theta = \mathrm{id}$ (Theorem 3.6.11, Corollary 3.6.12), we have*

$$\mathrm{tr}_A(\sigma) = \mathrm{tr}_A(\mathrm{id}_A) = \dim(A). \tag{3.131}$$

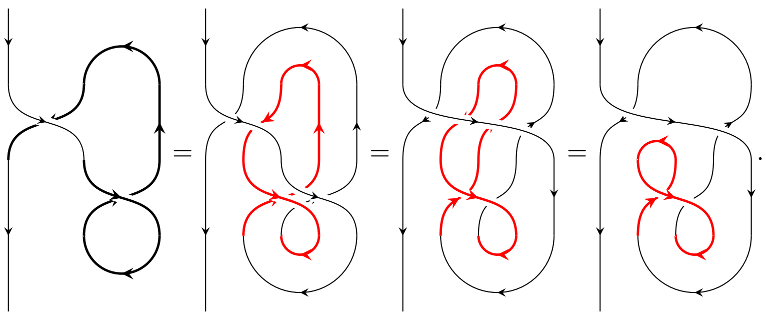
Proof. Lemma 3.6.9 implies that $\omega_{A^{\otimes(n+1)}} = \omega_A \otimes \omega_{A^{\otimes n}}$. We thus find

$\mathrm{tr}(\sigma) =$  . (3.132)

Decomposing

 (3.133)

we find

 (3.134)

Lemma 3.6.15 now shows that the statement holds. ■

Corollary 3.6.17. *Let $\mathcal{C}$ be a rigid symmetric monoidal category, let $n \geq 1$, and let $A \in \mathrm{Ob}(\mathcal{C})$. Let $\sigma \in S_n$ be a permutation of cycle type $c_1^{m_1} c_2^{m_2} \cdots c_d^{m_d}$, meaning that σ can be written as a product of m_i disjoint cycles of length c_i for each i, where the cycles are pairwise disjoint and $\sum_{k=1}^{d} c_k m_k = n$. For the induced morphism $\sigma : A^{\otimes n} \to A^{\otimes n}$, we have*

$$\mathrm{tr}(\sigma) = \otimes_{k=1}^{d} \otimes_{\ell=1}^{m_k} \dim(A). \tag{3.135}$$

Proof. We know that $\sigma = \otimes_{k=1}^{d} \otimes_{\ell=1}^{m_k} \sigma_{c_k}^{\ell}$, where $\sigma_{c_k}^{\ell}$ is a cycle of length c_k. Using $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \otimes \mathrm{tr}(g)$ (Proposition 3.5.4) and the above Proposition 3.6.16, we then see that

$$\mathrm{tr}_{A^{\otimes n}}(\sigma) = \otimes_{k=1}^{d} \otimes_{\ell=1}^{m_k} \mathrm{tr}_{A^{\otimes c_k}}(\sigma_{c_k}^{\ell}) = \otimes_{k=1}^{d} \otimes_{\ell=1}^{m_k} \dim(A). \tag{3.136}$$

■

4 Tensor Categories

At this point, we have all the necessary components in place to define the categories we are truly interested in: tensor categories. In this chapter, we will show how tensor categories naturally combine the structure of abelian categories and monoidal categories with duals.

4.1 Endomorphisms on the monoidal unit

Before introducing special types of categories combining abelian and monoidal structures, we first observe that the endomorphism monoid of the monoidal unit exhibits particularly nice behaviour, and that the category is naturally enriched over this monoid. This enrichment becomes especially interesting when the category is additionally pre-additive or abelian, which is the context we will consider in this chapter.

Proposition 4.1.1. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. The composition turns $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ into a commutative monoid, and for any two $f, g \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ we can write the composition as*

$$f \circ g = \lambda_{\mathbb{1}} \circ (f \otimes g) \circ \lambda_{\mathbb{1}}^{-1}. \tag{4.1}$$

If, in addition, $\mathcal{C}$ is pre-additive, this implies that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a commutative ring. If $\mathcal{C}$ is abelian and $\mathbb{1}$ is simple, then we find that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field.

Proof. Using the fact that the left and right unitors λ and ρ are natural isomorphisms ((3.2) and (3.3)), and $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$, we find

$$\begin{aligned} f \circ g &= f \circ \lambda_{\mathbb{1}} \circ (\mathrm{id}_{\mathbb{1}} \otimes g) \circ \lambda_{\mathbb{1}}^{-1} \\ &= \lambda_{\mathbb{1}} \circ (f \otimes \mathrm{id}_{\mathbb{1}}) \circ (\mathrm{id}_{\mathbb{1}} \otimes g) \circ \lambda_{\mathbb{1}}^{-1}. \\ &= \lambda_{\mathbb{1}} \circ (f \otimes g) \circ \lambda_{\mathbb{1}}^{-1} \end{aligned} \tag{4.2}$$

Using the same properties, we find

$$\begin{aligned} f \circ g &= \lambda_{\mathbb{1}} \circ (f \otimes g) \circ \lambda_{\mathbb{1}}^{-1} \\ &= \lambda_{\mathbb{1}} \circ (\mathrm{id}_{\mathbb{1}} \otimes g) \circ (f \otimes \mathrm{id}_{\mathbb{1}}) \circ \lambda_{\mathbb{1}}^{-1} \\ &= g \circ \lambda_{\mathbb{1}} \circ \lambda_{\mathbb{1}}^{-1} \circ f \\ &= g \circ f \end{aligned}. \tag{4.3}$$

The final statement follows from Schur's lemma 2.4.9. ■

Next, we will show that the category is enriched over this commutative monoid.

Proposition 4.1.2. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. There exist left and right actions of $(\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}), \circ)$ on all the hom-sets. For every $A, B \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), \zeta \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, we define*

$$\begin{aligned} \zeta \cdot f &:= \quad A \xrightarrow{\lambda_A^{-1}} \mathbb{1} \otimes A \xrightarrow{\zeta \otimes f} \mathbb{1} \otimes B \xrightarrow{\lambda_B} B \ , \\ f \cdot \zeta &:= \quad A \xrightarrow{\rho_A^{-1}} A \otimes \mathbb{1} \xrightarrow{f \otimes \zeta} B \otimes \mathbb{1} \xrightarrow{\rho_B} B \ . \end{aligned} \tag{4.4}$$

Proposition 4.1.1 shows that these actions coincide with the composition on $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$.

1. *If there exists a braiding on this category, then these two actions coincide.*

2. *The above maps define actions in the sense that, for any* $A, B \in \mathrm{Ob}(\mathcal{C})$ *and for all* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\zeta, \xi \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, *we have*

$$\begin{aligned} (\zeta \circ \xi) \cdot f &= \zeta \cdot (\xi \cdot f), \\ f \cdot (\zeta \circ \xi) &= (f \cdot \zeta) \cdot \xi. \end{aligned} \tag{4.5}$$

3. *The composition is linear with respect to these actions; more precisely, for all* $A, B, C \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\zeta \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, *we have*

$$\begin{aligned} \zeta \cdot (f \circ g) &= (\zeta \cdot f) \circ g = f \circ (\zeta \cdot g), \\ (f \circ g) \cdot \zeta &= (f \cdot \zeta) \circ g = f \circ (g \cdot \zeta). \end{aligned} \tag{4.6}$$

4. *These actions are compatible with each other, meaning that for any* $A, B \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\zeta, \xi \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, *we have*

$$(\zeta \cdot f) \cdot \xi = \zeta \cdot (f \cdot \xi). \tag{4.7}$$

5. *These actions are also compatible with the monoidal product, meaning that for any* $A, B, X, Y \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, *and* $\zeta \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, *we have*

$$\begin{aligned} (f \cdot \zeta) \otimes g &= f \otimes (\zeta \cdot g), \\ (\zeta \cdot f) \otimes g &= \zeta \cdot (f \otimes g), \\ f \otimes (g \cdot \zeta) &= (f \otimes g) \cdot \zeta. \end{aligned} \tag{4.8}$$

6. *If the monoidal product is linear in both arguments with respect to one of the actions (that is, if either* $f \otimes (\zeta \cdot g) = \zeta \cdot (f \otimes g)$ *always holds, or* $(f \cdot \zeta) \otimes g = (f \otimes g) \cdot \zeta$ *always holds), then the two actions coincide.*

7. *If* $\mathcal{C}$ *is pre-additive, then every hom-set becomes a* $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$*-bimodule.*

Proof. Note that, using the fact that the left and right unitors are natural isomorphisms ((3.2) and (3.3)), we can rewrite the actions as

$$\begin{aligned} \zeta \cdot f &= \lambda_B \circ (\zeta \otimes \mathrm{id}_B) \circ \lambda_B^{-1} \circ f = f \circ \lambda_A \circ (\zeta \otimes \mathrm{id}_A) \circ \lambda_A^{-1}, \\ f \cdot \zeta &= \rho_B \circ (\mathrm{id}_B \otimes \zeta) \circ \rho_B^{-1} \circ f = f \circ \rho_A \circ (\mathrm{id}_A \otimes \zeta) \circ \rho_A^{-1}. \end{aligned} \tag{4.9}$$

1. Let γ be a braiding on this category. Using Lemma 3.6.5 and the naturality of the braiding (3.76), we find

$$\begin{aligned} \rho_B^{-1} \circ \lambda_B \circ (\zeta \otimes \mathrm{id}_B) \circ \lambda_B^{-1} \circ \rho_B &= \gamma_{(\mathbb{1},B)} \circ (\zeta \otimes \mathrm{id}_B) \circ \gamma_{(B,\mathbb{1})} \\ &= (\mathrm{id}_B \otimes \zeta) \circ \gamma_{(\mathbb{1},B)} \circ \gamma_{(B,\mathbb{1})}. \\ &= \mathrm{id}_B \otimes \zeta \end{aligned} \tag{4.10}$$

   Using (4.9), we then conclude that $\zeta \cdot - = - \cdot \zeta$.

2. This follows from (4.9).

3. This also follows from (4.9).

4. Using the fact that the left unitor is a natural isomorphism (3.2), and functoriality of the monoidal product, we find

$$\begin{aligned} (\lambda_A \circ (\zeta \otimes \mathrm{id}_A) \circ \lambda_A^{-1}) \circ (\rho_A \circ (\mathrm{id}_A \otimes \xi) \circ \rho_A^{-1}) &= \lambda_A \circ (\zeta \otimes \mathrm{id}_A) \circ (\mathrm{id}_{\mathbb{1}} \otimes (\rho_A \circ \mathrm{id}_A \otimes \xi \circ \rho_A^{-1})) \circ \lambda_A^{-1} \\ &= \lambda_A \circ (\mathrm{id}_{\mathbb{1}} \otimes (\rho_A \circ \mathrm{id}_A \otimes \xi \circ \rho_A^{-1})) \circ (\zeta \otimes \mathrm{id}_A) \circ \lambda_A^{-1}. \\ &= (\rho_A \circ (\mathrm{id}_A \otimes \xi) \circ \rho_A^{-1}) \circ (\lambda_A \circ (\zeta \otimes \mathrm{id}_A) \circ \lambda_A^{-1}) \end{aligned} \tag{4.11}$$

   Applying this to (4.9), we conclude that the actions are compatible.

5. Using the naturality of the associator (3.1), the triangly identity (3.5), and functoriality of the monoidal product, we find

$$\begin{aligned} f \otimes (\zeta \cdot g) &= (\mathrm{id}_B \otimes \lambda_Y) \circ (f \otimes (\mathrm{id}_{\mathbb{1}} \otimes \mathrm{id}_Y)) \circ (\mathrm{id}_A \otimes (\zeta \otimes g)) \circ (\mathrm{id}_A \otimes \lambda_X^{-1}) \\ &= (\mathrm{id}_B \otimes \lambda_Y) \circ (f \otimes (\mathrm{id}_{\mathbb{1}} \otimes \mathrm{id}_Y)) \circ (\mathrm{id}_A \otimes (\zeta \otimes g)) \circ \alpha_{(A,\mathbb{1},X)} \circ (\rho_A \otimes \mathrm{id}_X) \\ &= (\mathrm{id}_B \otimes \lambda_Y) \circ (f \otimes (\mathrm{id}_{\mathbb{1}} \otimes \mathrm{id}_Y)) \circ \alpha_{(A,\mathbb{1},Y)} \circ ((\mathrm{id}_A \otimes \zeta) \otimes g) \circ (\rho_A \otimes \mathrm{id}_X) \\ &= (\mathrm{id}_B \otimes \lambda_Y) \circ \alpha_{(B,\mathbb{1},Y)} \circ ((f \otimes \mathrm{id}_{\mathbb{1}}) \otimes \mathrm{id}_Y) \circ ((\mathrm{id}_A \otimes \zeta) \otimes g) \circ (\rho_A \otimes \mathrm{id}_X) \\ &= (\rho_B \otimes \mathrm{id}_Y) \circ ((f \otimes \zeta) \otimes g) \circ (\rho_A \otimes \mathrm{id}_X) \\ &= (f \cdot \zeta) \otimes g \end{aligned} . \tag{4.12}$$

Using the naturality of the associator (3.1) and the identity $\lambda_A \otimes \mathrm{id}_X = \lambda_{A\otimes X} \circ \alpha_{\mathbb{1},A,X}$ (an alternative form of the triangle identity (3.5), see Lemma 2.5 on this nLab page, [aut25c], for a proof), we find

$$\begin{aligned} (\zeta \cdot f) \otimes g &= (f \otimes g) \circ (\lambda_A \otimes \mathrm{id}_X) \circ ((\zeta \otimes \mathrm{id}_A) \otimes \mathrm{id}_X) \circ (\lambda_A^{-1} \otimes \mathrm{id}_X) \\ &= (f \otimes g) \circ \lambda_{A\otimes X} \circ \alpha_{(\mathbb{1},A,X)} \circ ((\zeta \otimes \mathrm{id}_A) \otimes \mathrm{id}_X) \circ \alpha^{-1}_{(\mathbb{1},A,X)} \circ \lambda^{-1}_{A\otimes X} \\ &= (f \otimes g) \circ \lambda_{A\otimes X} \circ (\zeta \otimes (\mathrm{id}_{A\otimes X})) \circ \lambda^{-1}_{A\otimes X} \\ &= \zeta \cdot (f \otimes g) \end{aligned} . \tag{4.13}$$

Similarly, one proves

$$f \otimes (g \cdot \zeta) = (f \otimes g) \cdot \zeta. \tag{4.14}$$

6. Suppose that $(f \cdot \zeta) \otimes g = (f \otimes g) \cdot \zeta$ always holds. For any $f : A \to B, \zeta : \mathbb{1} \to \mathbb{1}$, we then find

$$\begin{aligned} \zeta \cdot f &= \lambda_B \circ (\mathrm{id}_{\mathbb{1}} \otimes (\zeta \cdot f)) \circ \lambda_A^{-1} \\ &= \lambda_B \circ ((\mathrm{id}_{\mathbb{1}} \cdot \zeta) \otimes f) \circ \lambda_A^{-1} \\ &= \lambda_B \circ ((\mathrm{id}_{\mathbb{1}} \otimes f) \cdot \zeta) \circ \lambda_A^{-1}. \\ &= (\lambda_B \circ (\mathrm{id}_{\mathbb{1}} \otimes f) \circ \lambda_A^{-1}) \cdot \zeta \\ &= f \cdot \zeta \end{aligned} \tag{4.15}$$

7. This is trivial from all of the above.

■

4.2 Multiring and ring categories

First, we introduce categories equipped with both a monoidal and an abelian structure, which we require to be compatible in a suitable sense. It is clear what this compatibility should entail: the monoidal product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ becomes a functor between two abelian categories, and should be a morphism in the category of abelian categories, or thus an exact functor. This means that:

1. the monoidal product $\otimes$ is bilinear on morphisms, i.e. $(f_1 + f_2) \otimes (g_1 + g_2) = f_1 \otimes g_1 + f_1 \otimes g_2 + f_2 \otimes g_1 + f_2 \otimes g_2$,
2. the monoidal product $\otimes$ is biexact, i.e. exact in both arguments[1].

Bilinearity of the monoidal product has the following important corollary.

[1]Actually, exactness of the bifunctor would just mean that $\otimes$ maps short exact sequences in $\mathcal{C} \times \mathcal{C}$ to short exact sequences in $\mathcal{C}$. However, the non-zero part of short exact sequences in $\mathcal{C} \times \mathcal{C}$ can be written as the composition of a short exact sequence in the first argument (with an identity morphism in the second), and a short exact sequence in the second argument (with an identity morphism in the first). Functoriality of the monoidal product then shows that it is enough to have exactness in both arguments to obtain exactness on $\mathcal{C} \times \mathcal{C}$.

Proposition 4.2.1. *Let $\mathcal{C}$ be an additive monoidal category such that the monoidal product is bilinear on morphisms, and let $A, B, X, Y \in \mathrm{Ob}(\mathcal{C})$. We have*

$$(A \oplus B) \otimes (X \oplus Y) = (A \otimes X) \oplus (A \otimes Y) \oplus (B \otimes X) \oplus (B \otimes Y) \tag{4.16}$$

with $\mathrm{inc}_{U \otimes V} = \mathrm{inc}_U \otimes \mathrm{inc}_V$ *and* $\mathrm{proj}_{U \otimes V} = \mathrm{proj}_U \otimes \mathrm{proj}_V$ *for* $U = A, B$ *and* $V = X, Y$.

Proof. This follows from the functoriality and linearity of the monoidal product. For example: $(\mathrm{proj}_A \otimes \mathrm{proj}_X) \circ (\mathrm{inc}_A \otimes \mathrm{inc}_X) = (\mathrm{proj}_A \circ \mathrm{inc}_A) \otimes (\mathrm{proj}_X \circ \mathrm{inc}_X) = \mathrm{id}_A \otimes \mathrm{id}_X = \mathrm{id}_{A \otimes X}$ through functoriality, and

$$\begin{aligned}\sum_{\substack{U=A,B\\V=X,Y}} (\mathrm{inc}_U \otimes \mathrm{inc}_V) \circ (\mathrm{proj}_U \otimes \mathrm{proj}_V) &= (\mathrm{inc}_A \circ \mathrm{proj}_A + \mathrm{inc}_B \circ \mathrm{proj}_B) \otimes (\mathrm{inc}_X \circ \mathrm{proj}_X + \mathrm{inc}_Y \circ \mathrm{proj}_Y) \\ &= \mathrm{id}_{A \oplus B} \otimes \mathrm{id}_{X \oplus Y} \\ &= \mathrm{id}_{(A \oplus B) \otimes (X \oplus Y)}\end{aligned} . \tag{4.17}$$

■

Corollary 4.2.2. *Let $\mathcal{C}$ be an additive monoidal category equipped with a braiding γ, such that the monoidal product is bilinear on morphisms. For any $A, B, X, Y \in \mathrm{Ob}(\mathcal{C})$, we have*

$$\gamma_{(A \oplus B, X \oplus Y)} = \sum_{\substack{U=A,B\\V=X,Y}} \mathrm{inc}_{V \otimes U} \circ \gamma_{(U,V)} \circ \mathrm{proj}_{U \otimes V} . \tag{4.18}$$

Proof. We have

$$\begin{aligned}\gamma_{(A \oplus B, X \oplus Y)} &= \gamma_{(A \oplus B, X \oplus Y)} \circ (\mathrm{inc}_A \circ \mathrm{proj}_A + \mathrm{inc}_B \circ \mathrm{proj}_B) \otimes (\mathrm{inc}_X \circ \mathrm{proj}_X + \mathrm{inc}_Y \circ \mathrm{proj}_Y) \\ &= \gamma_{(A \oplus B, X \oplus Y)} \circ \left(\sum_{\substack{U=A,B\\V=X,Y}} (\mathrm{inc}_U \otimes \mathrm{inc}_V) \circ (\mathrm{proj}_U \otimes \mathrm{proj}_V) \right) \\ &= \sum_{\substack{U=A,B\\V=X,Y}} \mathrm{inc}_{V \otimes U} \circ \gamma_{(U,V)} \circ \mathrm{proj}_{U \otimes V}\end{aligned} . \tag{4.19}$$

■

Proposition 4.1.1 and Proposition 4.1.2 now show that the category is enriched over the commutative ring $R = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$. If we want the category to behave in a reasonable and manageable way (for example, think of Example 30), we additionally want the hom-spaces to be finitely generated and projective over this ring. If the category is enriched over a field, for example when the monoidal unit is simple, this leads to the following notion of locally finite categories.

Definition 4.2.3 (Locally finite categories, [EGNO15, Definition 1.8.1]). Let $\mathcal{C}$ be a $\mathbb{K}$-linear abelian category for some field $\mathbb{K}$. It is called *locally finite* if

1. every object is of finite length,
2. all the hom-spaces are finite-dimensional (i.e. $\mathcal{C}$ is $\mathbf{FinVect}_{\mathbb{K}}$-enriched).

Definition 4.2.4 (Multiring and ring categories, [EGNO15, Definition 4.2.3]). Let $\mathbb{K}$ be a field. A locally finite $\mathbb{K}$-linear abelian monoidal category is called a *multiring category* if the monoidal product is bilinear and biexact. If, in addition, the monoidal unit $\mathbb{1}$ is such that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \cong \mathbb{K}$ as algebras, then it is called a *ring category.*

4.3 Multitensor and tensor categories

4.3.1 Introducing duals into the mix

Second, we want to introduce duals into the mix. Compatibility then asks the dualisation functors to be exact. If the category is equipped with both left and right duals, Remark 3.4.9 shows that the dualisation functors are exact. However, we will now show that we don't need both left and right duals; having just one kind of dual is enough.

Proposition 4.3.1 ([EGNO15, Proposition 4.2.9]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and abelian category such that the monoidal product is bilinear and biexact. The left (resp. right) dualisation functor is exact.*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. We will use the fact that the hom-functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X)$ reflect left exact sequences (Proposition 1.2.4). This implies that

$$\begin{gathered} 0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \quad \text{and} \\ C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \longrightarrow 0 \end{gathered} \tag{4.20}$$

are exact if and only if

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, C^*) \xrightarrow{g^* \circ -} \mathrm{Hom}_{\mathcal{C}}(X, B^*) \xrightarrow{f^* \circ -} \mathrm{Hom}_{\mathcal{C}}(X, A^*) \quad \text{and} \\ 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A^*, X) \xrightarrow{- \circ f^*} \mathrm{Hom}_{\mathcal{C}}(B^*, X) \xrightarrow{- \circ g^*} \mathrm{Hom}_{\mathcal{C}}(C^*, X) \end{gathered} \tag{4.21}$$

are exact.

We will only prove the first of these statements. We can rewrite the sequence as

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X \otimes C, \mathbb{1}) \xrightarrow{- \circ (\mathrm{id}_X \otimes g)} \mathrm{Hom}_{\mathcal{C}}(X \otimes B, \mathbb{1}) \xrightarrow{- \circ (\mathrm{id}_X \otimes f)} \mathrm{Hom}_{\mathcal{C}}(X \otimes A, \mathbb{1}) \ . \tag{4.22}$$

This sequence is exact because we are applying a left exact contravariant functor (the second argument of the monoidal product) to a right exact sequence[2]. ■

A second noteworthy result in this setting is that the monoidal product of any object with a projective object is again projective.

Proposition 4.3.2 ([EGNO15, Proposition 4.2.12]). *Let $\mathcal{C}$ be a monoidal and abelian category such that the monoidal product is bilinear and right (resp. left) exact in the first (resp. second) argument, and let $A \in \mathrm{Ob}(\mathcal{C})$ be projective. If an object $X \in \mathrm{Ob}(\mathcal{C})$ has a left (resp. right) dual X^*, then the object $A \otimes X$ (resp. $X \otimes A$) is projective.*

Proof. We want to show that $\mathrm{Hom}_{\mathcal{C}}(A \otimes X, -)$ is right exact, or equivalently (through Proposition 3.4.5) that $\mathrm{Hom}_{\mathcal{C}}(A, - \otimes X^*)$ is right exact. We know that $- \otimes X^*$ and $\mathrm{Hom}_{\mathcal{C}}(A, -)$ are right exact, which shows that their composition $\mathrm{Hom}_{\mathcal{C}}(A, - \otimes X^*)$ is right exact too. ■

Remarkably, biexactness of the monoidal product follows from bilinearity if we require our categories to be rigid.

Proposition 4.3.3. *Let $\mathcal{C}$ be an abelian rigid (resp. left rigid, right rigid) monoidal category with a bilinear monoidal product. The monoidal product is then biexact (resp. right, left exact in the first argument, and left, right exact in the second argument).*

Proof. Proposition 3.4.5 shows that we have adjoint pairs $(A^* \otimes -, A \otimes -), (A \otimes -, {}^*A \otimes -), (- \otimes A, - \otimes A^*), (- \otimes {}^*A, - \otimes A)$. Theorem 1.4.3 then implies the result. ■

[2] Note that it is not sufficient to ask only for left or right duals, without the additional biexactness condition on the monoidal product. Indeed, we would then find that $X \otimes -$ is left, right exact but not necessarily right, left exact.

4.3.2 Properties of the monoidal unit

Corollary 4.3.4 ([EGNO15, Corollary 4.2.13]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and abelian category such that the monoidal product is bilinear. The monoidal unit is projective if and only if the category is semisimple.*

Proof. Suppose that $\mathbb{1}$ is projective, then any object $X \cong \mathbb{1} \otimes X$ is projective due to the above Proposition 4.3.2 and Proposition 4.3.3. Proposition 2.3.9 then implies that $\mathcal{C}$ is semisimple. ■

Remark 4.3.5. The above Corollary 4.3.4 is a generalisation of Maschke's theorem, as explained in [EGNO15, Remark 4.2.14].

We will now show that the monoidal unit, although not projective, behaves nicely.

Proposition 4.3.6 ([DM82, Proposition 1.17]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and abelian category such that the monoidal product is bilinear and biexact. If $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field, then the monoidal unit is simple.*

Proof. Let (A, f) be any subobject of $\mathbb{1}$ (we assume that $\mathbb{1} \ncong 0$). We extend this monomorphism $f : A \to \mathbb{1}$ to a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} \mathbb{1} \xrightarrow{g} B \longrightarrow 0 \ . \tag{4.23}$$

As $- \otimes A$ is exact, and the right unitor is a natural isomorphism (3.3), we find the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes A & \xrightarrow{f \otimes \mathrm{id}_A} & \mathbb{1} \otimes A & \xrightarrow{g \otimes \mathrm{id}_A} & B \otimes A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_A \otimes f} & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{1}} \otimes f} & & \downarrow{\scriptstyle \mathrm{id}_B \otimes f} & & \\ 0 & \longrightarrow & A \otimes \mathbb{1} & \xrightarrow{f \otimes \mathrm{id}_{\mathbb{1}}} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{g \otimes \mathrm{id}_{\mathbb{1}}} & B \otimes \mathbb{1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \rho_{\mathbb{1}}} & & \downarrow{\scriptstyle \rho_B} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & \mathbb{1} & \xrightarrow{g} & B & \longrightarrow & 0 \end{array} \ . \tag{4.24}$$

As $g \circ f = 0$, and $\rho_B \circ (\mathrm{id}_B \otimes f)$ is a monomorphism (because $B \otimes -$ is exact), we find that $g \otimes \mathrm{id}_A = 0$, hence that $B \otimes A \cong 0$ because $g \otimes \mathrm{id}_A$ is an epimorphism. This then implies that $\rho_A \circ (f \otimes \mathrm{id}_A) : A \otimes A \to A$ is an isomorphism.

As the left dualisation functor is exact (Proposition 4.3.1), we know that

$$(\mathrm{Ker}(f^*), \ker(f^*)) = (B^*, g^*). \tag{4.25}$$

As $B \otimes -$ is exact too, we find that

$$(\mathrm{Ker}(\mathrm{id}_B \otimes f^*), \ker(\mathrm{id}_B \otimes f^*)) = (B \otimes B^*, \mathrm{id}_B \otimes g^*). \tag{4.26}$$

Because $\mathrm{Hom}_{\mathcal{C}}(B \otimes A, B) \cong \mathrm{Hom}_{\mathcal{C}}(B, B \otimes A^*)$ (Proposition 3.4.5), we find that $B \otimes A = 0$ implies that $\mathrm{id}_B \otimes f^* = 0$. This then implies that $(\mathrm{Ker}(\mathrm{id}_B \otimes f^*), \ker(\mathrm{id}_B \otimes f^*)) \cong (B, \mathrm{id}_B)$. The unique isomorphism between these kernels is $\rho_B \circ (\mathrm{id}_B \otimes g^*) : B \otimes B^* \to B$.

The fact that $\mathrm{id}_B \otimes g^*$ is an isomorphism then implies that $B \otimes A^* = \mathrm{Coker}(\mathrm{id}_B \otimes g^*) \cong 0$. As the monoidal product is bilinear, this implies that $A \otimes B^* \cong 0$.

$- \otimes B^*$ is exact, which implies that we also have the short exact sequence

$$0 \longrightarrow A \otimes B^* \xrightarrow{f \otimes \mathrm{id}_{B^*}} \mathbb{1} \otimes B^* \xrightarrow{g \otimes \mathrm{id}_{B^*}} B \otimes B^* \longrightarrow 0 \ . \tag{4.27}$$

The facts that $A \otimes B^* \cong 0$ and that $\rho_B \circ (\mathrm{id}_B \otimes g^*) : B \otimes B^* \to B$ is an isomorphism then imply that we obtain an isomorphism

$$0 \longrightarrow B^* \xrightarrow{\rho_B \circ (g \otimes g^*) \circ \lambda_{B^*}^{-1} = g \circ g^*} B \longrightarrow 0 \ . \tag{4.28}$$

We conclude that there is a short exact sequence

$$0 \longrightarrow A \xrightarrow{\quad f \quad} \mathbb{1} \xrightarrow{(g\circ g^*)^{-1}\circ g} B^* \longrightarrow 0 \ . \tag{4.29}$$

This short exact sequence is split as $(g \circ g^*)^{-1} \circ g : \mathbb{1} \to B^*$ has the right inverse g^*. We conclude that

$$\mathbb{1} = A \oplus B^*. \tag{4.30}$$

This implies that the endomorphism ring $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ decomposes as a direct sum

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \cong \mathrm{Hom}_{\mathcal{C}}(A, A) \oplus \mathrm{Hom}_{\mathcal{C}}(A, B^*) \oplus \mathrm{Hom}_{\mathcal{C}}(B^*, A) \oplus \mathrm{Hom}_{\mathcal{C}}(B^*, B^*)\,, \tag{4.31}$$

and if $A \ncong 0$ and $B^* \ncong 0$, then this is at least two dimensional as a vector space over $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$. We conclude that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ can only be a field if $A \cong 0$ or $B \cong 0$, which implies that f is either zero or an isomorphism. This implies that $\mathbb{1}$ is simple. ■

Remark 4.3.7. The above proposition is the converse to Proposition 4.1.1.

The above proposition allows us to prove a statement that is clearly true for finitely generated projective modules: the evaluation is an epimorphism and the coevaluation is a monomorphism.

Corollary 4.3.8 ([EGNO15, Corollary 4.3.9]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and abelian category such that the monoidal product is bilinear and biexact. If $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field, then all evaluation morphisms are epimorphisms and all coevaluation morphisms are monomorphisms.*

Proof. This follows from Lemma 2.2.10 and Proposition 4.3.6. ■

Corollary 4.3.9 ([DM82, Proposition 1.19] and [EGNO15, Remark 4.3.10]). *Let $\mathcal{C}, \mathcal{D}$ be left (resp. right) rigid monoidal and abelian categories such that the monoidal product is bilinear and biexact. If $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ is a field, then any exact monoidal functor $\mathcal{C} \to \mathcal{D}$ is faithful (if $\mathcal{D}$ contains a non-zero object).*

Proof. Suppose that $A \in \mathrm{Ob}\,(\mathcal{C})$ is not a null object. F is exact, which implies that F maps the monomorphism coev_A (due to Corollary 4.3.8) to a monomorphism. As F is monoidal, we also have $F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}}$. This implies that we have a monomorphism from a non-zero object to $F(A)$, hence that $F(A)$ is not a null object.

Let f be any morphism in $\mathcal{C}$, and suppose that $f \neq 0$, which implies that $\mathrm{Im}(f)$ is not a null object. Applying the above to $\mathrm{Im}(f)$ implies that $F(\mathrm{Im}(f))$ is not a null object, or thus that $F(f) \neq 0$. ■

4.3.3 Tensor categories

When introducing duals to multiring categories, Proposition 4.3.3 shows that we can ignore the biexactness condition for the monoidal product. This leads to the following natural definition of categories mixing an abelian structure, and a monoidal structure with duals.

Definition 4.3.10 (Multitensor and tensor categories, [EGNO15, Definition 4.1.1]). Let $\mathbb{K}$ be a field. A $\mathbb{K}$-linear abelian rigid monoidal category is called a *multitensor category* if the monoidal product is bilinear on morphisms. If, in addition, the monoidal unit $\mathbb{1}$ is such that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \cong \mathbb{K}$ as algebras, then it is called a *tensor category*.

Remark 4.3.11. We did not assume our tensor categories to be locally finite. This is because we are following [DM82; Del02; Del90] here. It should be noted that [EGNO15, Definition 4.1.1] defines tensor categories to be locally finite. It can be shown that in a tensor category in which all objects have finite length, all hom-spaces are automatically finite-dimensional (see [Del02, Proposition 1.1]). It is thus sufficient to assume that all objects have finite length to obtain locally finite categories.

Definition 4.3.12 (Tensor subcategories, [EGNO15, Definition 4.11.1]). Let $\mathcal{C}$ be a multitensor category. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a *tensor subcategory* if it contains the monoidal unit and is closed under

1. (finite) direct sums,
2. subquotients (i.e. quotients of subobjects),
3. monoidal products,
4. duals.

We say that a tensor subcategory $\mathcal{D} \subseteq \mathcal{C}$ is finitely generated if it is generated by a single object $A \in \mathrm{Ob}(\mathcal{C})$, i.e. if every object can be obtained from A by iterating direct sums, tensor products, dualisation, and taking subquotients.

Finally, the appropriate notion of structure-preserving functors between (multi)tensor categories (and also multiring categories) is that of exact monoidal functors.

Definition 4.3.13 (Tensor functors, [EGNO15, Definition 4.2.5]). Let $\mathcal{C}, \mathcal{D}$ be two multitensor categories. A *tensor functor* is an exact monoidal functor $\mathcal{C} \to \mathcal{D}$.

Example 39. Let $\mathbb{K}$ be a field, the category of finite-dimensional $\mathbb{K}$-vector spaces $\mathbf{FinVect}_{\mathbb{K}}$ is a tensor category.

Example 40. Let G be a group and let $\mathbb{K}$ be a field. The category of finite-dimensional $\mathbb{K}$-linear G-representations $\mathbf{FinRep}_{\mathbb{K}}(G)$ is abelian (Example 19) and rigid monoidal (Examples 25 and 32). Furthermore, it is locally finite and $\mathbb{K}$-linear, and is such that tensor product is bilinear on morphisms. This implies that it is a multitensor category. As the monoidal unit $(\mathbb{K}, g \mapsto \mathrm{id}_{\mathbb{K}})$ has a one-dimensional hom-space (which is inherited from $\mathbf{FinVect}_{\mathbb{K}}$), we conclude that it is a tensor category.

4.4 Multifusion and fusion categories

Given the complexity of multitensor categories in general, it is natural to introduce a particularly simple subclass characterised by involving only finitely many pieces of data[3].

Definition 4.4.1 (Multifusion and fusion categories, [EGNO15, Definition 4.1.1]). A multitensor category is called a *multifusion category* if it has a finite amount of simple objects and it is semisimple. If the underlying multitensor category is actually a tensor category, then we call it a *fusion category*.

Note that a tensor subcategory $\mathcal{D}$ of a (multi)fusion category $\mathcal{C}$, also called a *fusion subcategory*, is a full subcategory containing the monoidal unit, closed under duals, monoidal products, and finite direct sums, and is such that every simple object appearing in a decomposition of an object of $\mathcal{D}$ is also contained in $\mathcal{D}$.

4.5 Symmetric tensor categories and a look at the literature

4.5.1 Introducing braidings into the mix

In what follows we will often be interested in symmetric tensor categories, i.e. tensor categories that are equipped with a symmetric braiding. There are a few reasons for this:

1. Theorem 3.6.11 and Corollary 3.6.12 imply that symmetric tensor categories are pivotal and spherical. This implies that these categories are amenable to semisimplification via negligible morphisms, as will be discussed in Chapter 5.
2. Symmetric tensor categories provide a natural setting for many familiar constructions in algebra; we will return to this shortly and explore it further in Chapter 6.
3. Symmetric tensor categories are relatively well understood. This will also be discussed shortly.

[3] This can be taken quite literally: by using skeletal data, multifusion categories can be encoded using only a finite amount of scalars, as discussed in my literature study [Sle24].

Remark 4.5.1. For many algebraic objects in locally finite symmetric tensor categories $\mathcal{C}$, one is often interested in both objects in $\mathcal{C}$ (the "finite-dimensional" objects) and objects in $\mathcal{C}^{\text{ind}}$ (the "possibly infinite-dimensional" objects). It is therefore important to note that the ind-cocompletion of a symmetric tensor category is itself a symmetric tensor category: all constructions and properties of symmetric tensor categories extend through filtered colimits. In particular, since the monoidal product preserves colimits, it naturally extends to the ind-cocompletion.

The correct notion of structure-preserving functors for braided tensor categories is just the combination of a braided monoidal functor with a tensor functor.

Definition 4.5.2 (Braided tensor functor, [EGNO15, Definition 8.1.7]). Let $\mathcal{C}, \mathcal{D}$ be braided multitensor categories. An exact braided monoidal functor is called a *braided tensor functor*. If the braiding is symmetric, then a braided tensor functor is called a *symmetric tensor functor*.

4.5.2 Symmetric and exterior powers of objects

Theorem 3.6.14 shows that there is a natural action of S_n on any n-fold tensor product of an object in a symmetric multitensor category.

Interestingly, this action allows us to show that dimensions of objects in symmetric tensor categories are well-behaved.

Proposition 4.5.3 ([EGNO15, Exercise 9.9.9 (ii)]). *Let $\mathcal{C}$ be a symmetric tensor category, and let $A \in \operatorname{Ob}(\mathcal{C})$. If $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field of positive characteristic $p > 0$, then*

$$\dim(A) \in \mathbb{F}_p. \tag{4.32}$$

Proof. Let $\sigma : A^{\otimes p} \to A^{\otimes p}$ be the cyclic permutation used in Proposition 3.6.16, and define $a := \operatorname{id}_{A^{\otimes p}} - \sigma$. Note that $a^p = 0$, which implies that $\operatorname{tr}(a) = 0$ (through a result we will prove later, Corollary 5.4.3).

However, we also have

$$\operatorname{tr}(\operatorname{id}_{A^{\otimes p}} - \sigma) = \operatorname{tr}(\operatorname{id}_{A^{\otimes p}}) - \operatorname{tr}(\sigma) = \dim(A)^p - \dim(A). \tag{4.33}$$

This implies that $\dim(A)$ is a fixpoint of the Frobenius endomorphism on $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, hence that $\dim(A) \in \mathbb{F}_p$. ■

The action of the symmetric group also allows us to define symmetric and exterior powers.

Definition 4.5.4 (Symmetric and exterior powers, [EGNO15, Definition 9.9.5]). Let $\mathcal{C}$ be a symmetric tensor category, let $A \in \operatorname{Ob}(\mathcal{C})$, and let $n \in \mathbb{N}$.

1. The n^{th} *symmetric power* of A, denoted $S^n(A)$ or S^nA, is the maximal quotient of $A^{\otimes n}$ on which the action of S_n is trivial.
2. The n^{th} *exterior power* of A, denoted $\wedge^n(A)$ or $\wedge^nA$, is the maximal quotient of $A^{\otimes n}$ on which the action of S_n factors through the sign representation.

This means that $(S^n(A), s^n(A) : A^{\otimes n} \to S^n(A))$ is the colimit of the (finite) diagram $(\sigma : A^{\otimes n} \to A^{\otimes n})_{\sigma \in S_n}$, and that $(\wedge^n(A), a^n(A) : A^{\otimes n} \to \wedge^n(A))$ is the colimit of the (finite) diagram $(\operatorname{sgn}(\sigma)\sigma : A^{\otimes n} \to A^{\otimes n})_{\sigma \in S_n}$ (i.e. a coequaliser of all these morphisms at the same time). Note that these colimits exist as $\mathcal{C}$ is abelian, hence finitely cocomplete.

Proposition 4.5.5. *Let $\mathcal{C}$ be a symmetric tensor category, and let $A \in \operatorname{Ob}(\mathcal{C})$. Let n be smaller than the characteristic of $\mathbb{K} = \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, or any natural number if the characteristic is zero. We define the* symmetriser $s_n : A^{\otimes n} \to A^{\otimes n}$ *and* skew-symmetriser $a_n : A^{\otimes n} \to A^{\otimes n}$ *as the action of $\frac{1}{n!}\sum_{\sigma \in S_n} \sigma$ and $\frac{1}{n!}\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma$ (in the group algebra $\mathbb{K}S_n$) on $A^{\otimes n}$ through Theorem 3.6.14.*

We have $(S^n(A), s^n(A)) = (\operatorname{Coim}(s_n), \operatorname{coim}(s_n))$ and $(\wedge^n(A), a^n(A)) = (\operatorname{Coim}(a_n), \operatorname{coim}(a_n))$.

Proof. As $s_n \circ \sigma = s_n$ and $a_n \circ \sigma = a_n$ for any $\sigma \in S_n$, we find that there are unique morphisms $f : S^n(A) \to \mathrm{Coim}(s^n)$ and $g : \wedge^n(A) \to \mathrm{Coim}(a^n)$ such that $f \circ s^n(A) = s_n$ and $g \circ a^n(A) = a_n$.

We also have $s^n(A) = s^n(A) \circ s_n$ and $a^n(A) = a^n(A) \circ a_n$, which shows that $s^n(A) = s^n(A) \circ s_n$ and $a^n(A) = a^n(A) \circ a_n$. This implies that there exist unique morphisms $\overline{f} : \mathrm{Coim}(s_n) \to S^n(A)$ and $\overline{g} : \mathrm{Coim}(a_n) \to \wedge^n(A)$ such that $s^n(A) = \overline{f} \circ \mathrm{coim}(s_n)$ and $a^n(A) = \overline{g} \circ \mathrm{coim}(a_n)$. Through the colimit properties, we then see that $\overline{f}$ and $\overline{g}$ are inverses to f and g respectively. ■

Corollary 4.5.6. *Let $\mathcal{C}$ be a symmetric tensor category over a field that is not of characteristic two, and let $A \in \mathrm{Ob}(\mathcal{C})$. Let $s_2 = \frac{1}{2}(\mathrm{id}_{A\otimes A} + \gamma_{(A,A)})$ and $a_2 = \frac{1}{2}(\mathrm{id}_{A\otimes A} - \gamma_{(A,A)})$ be the symmetriser and skew-symmetriser on $A \otimes A$ introduced above. The following sequences are split short exact*

$$\begin{gathered} 0 \longrightarrow \wedge^2 A \xrightarrow{\mathrm{im}(a_2)} A \otimes A \xrightarrow{\mathrm{coim}(s_2)} S^2 A \longrightarrow 0 \\ 0 \longrightarrow S^2 A \xrightarrow{\mathrm{im}(s_2)} A \otimes A \xrightarrow{\mathrm{coim}(a_2)} \wedge^2 A \longrightarrow 0 \end{gathered}. \tag{4.34}$$

As a corollary, we find

$$A \otimes A = S^2 A \oplus \wedge^2 A. \tag{4.35}$$

Proof. Note first that $a_2 \circ s_2 = 0$.

Suppose that $a_2 \circ k = 0$ for any morphism k. We then find $k = \gamma_{(A,A)} \circ k$, which shows that $s_2 \circ k = k$. Through the definition of $S^2 A$ and Proposition 4.5.5, this shows that there is a unique morphism $\overline{k}$ such that $\mathrm{im}(s_2) \circ \overline{k} = k$. This implies that $\mathrm{im}(s_2) = \ker(a_2) = \ker(\mathrm{coim}(a_2))$. Furthermore, $s_2 \circ s_2 = s_2$, hence $\mathrm{coim}(s_2) \circ \mathrm{im}(s_2) = \mathrm{id}_{S^2 A}$. This implies that $\mathrm{im}(s_2)$ is split. ■

More generally than the previous statement, we have the following.

Lemma 4.5.7. *Let $\mathcal{C}$ be a symmetric tensor category, and let $A \in \mathrm{Ob}(\mathcal{C})$. Let n be smaller than the characteristic of $\mathbb{K} = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, or any natural number if the characteristic is zero. Let s_n and a_n be the symmetriser and skew-symmetriser introduced in the above Proposition 4.5.5. We have*

$$\mathrm{coim}(s_n) \circ \mathrm{im}(s_n) = \mathrm{id}_{S^n A} \textit{ and } \mathrm{coim}(a_n) \circ \mathrm{im}(a_n) = \mathrm{id}_{\wedge^n A}. \tag{4.36}$$

In particular, $\mathrm{im}(s_n)$ and $\mathrm{im}(a_n)$ are split monomorphisms, and $\mathrm{coim}(s_n)$ and $\mathrm{coim}(a_n)$ are split epimorphisms.

Proof. We have $s_n \circ s_n = s_n$ and $a_n \circ a_n = a_n$. This implies that $\mathrm{im}(s_n) \circ \mathrm{coim}(s_n) \circ \mathrm{im}(s_n) \circ \mathrm{coim}(s_n) = \mathrm{im}(s_n) \circ \mathrm{coim}(s_n)$, hence that $\mathrm{coim}(s_n) \circ \mathrm{im}(s_n) = \mathrm{id}_{S^n A}$. Similarly, we show that $\mathrm{coim}(a_n) \circ \mathrm{im}(a_n) = \mathrm{id}_{\wedge^n A}$. ■

Proposition 4.5.8 ([EGNO15, Exercise 9.9.9 (i)]). *Let $\mathcal{C}$ be a symmetric tensor category, let $n \geq 1$, and let $A \in \mathrm{Ob}(\mathcal{C})$. We have (whenever this makes sense)*

$$\dim(S^n A) = \binom{\dim A + n - 1}{n} \textit{ and } \dim(\wedge^n A) = \binom{\dim A}{n}. \tag{4.37}$$

Proof. We claim that $\dim(S^n A) = \mathrm{tr}(s_n)$ and $\dim(\wedge^n A) = \mathrm{tr}(a_n)$, where s_n and a_n are the symmetriser and skew-symmetriser introduced in the above Proposition 4.5.5.

We have $\mathrm{coim}(s_n) \circ \mathrm{im}(s_n) = \mathrm{id}_{S^n A}$ and $\mathrm{coim}(a_n) \circ \mathrm{im}(a_n) = \mathrm{id}_{\wedge^n A}$ through Lemma 4.5.7. Lemma 3.5.8 then shows that $\dim(S^n A) = \mathrm{tr}_{S^n A}(\mathrm{id}_{S^n A}) = \mathrm{tr}_{S^n A}(\mathrm{coim}(s_n) \circ \mathrm{im}(s_n)) = \mathrm{tr}_{A^{\otimes n}}(\mathrm{im}(s_n) \circ \mathrm{coim}(s_n)) = \mathrm{tr}_{A^{\otimes n}}(s_n)$, and similarly $\dim(\wedge^n A) = \mathrm{tr}_{A^{\otimes n}}(a_n)$.

Using Corollary 3.6.17, we then find $\dim(S^n A) = p(\dim(A))$ and $\dim(\wedge^n A) = q(\dim(A))$ for some known polynomials p, q. As these are the same polynomials one obtains in $\mathbf{FinVect}_{\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathbb{1})}$, and we know the dimensions of symmetric and exterior powers in this category[4], we conclude the result. ■

[4]In positive characteristic $p > 0$, we have to consider the result modulo p.

4.5.3 The classification of pre-Tannakian symmetric tensor categories

Symmetric tensor categories provide the natural (and minimal) framework in which to carry out commutative algebra, algebraic geometry, representation theory, and so on. Indeed, we will see later that they are the appropriate setting for defining and studying commutative algebras and related structures. Although our primary interest in this text lies in non-associative algebras, we will nevertheless work within symmetric tensor categories, as their structure allows us to extract meaningful results (about their structure) even in the non-associative setting. For instance, Lie algebras inherently require a symmetric (or at least braided) tensor structure, and without such structure, general non-associative algebras tend to be hard to describe.

For this reason, and because symmetric tensor categories are considered inherently interesting, there has been an effort to classify symmetric tensor categories (over algebraically closed fields) over the past few decades. This programme was initiated by Pierre Deligne, and has since been developed further by researchers such as Victor Ostrik, Pavel Etingof, Kevin Coulembier, among others[5].

General symmetric tensor categories seem impossible to classify, and for this reason (among others) attention is typically restricted to those symmetric tensor categories in which all objects have finite length.

Definition 4.5.9 (Pre-Tannakian categories, [Del90, § 2.1] and [CEO24b, § 2.2.1]). A *pre-Tannakian category* is a symmetric tensor category in which all objects have finite length.

In the course of classifying pre-Tannakian symmetric tensor categories, the notion of categories with favourable polynomial noetherian and artinian properties has emerged as an important organising principle.

Definition 4.5.10 (Pre-Tannakian categories of moderate growth, [CEO24b, § 2.2.1]). A pre-Tannakian category $\mathcal{C}$ is called *of moderate* or *subexponential growth* if, for every object $A \in \mathrm{Ob}\,(\mathcal{C})$, there exists an integer $N_A \geq 0$ such that

$$\mathrm{len}(A^{\otimes n}) \leq N_A^n \text{ for all } n \geq 0. \tag{4.38}$$

We thus have (following [CEO24b, § 1.3]) three interesting (2-)categories of symmetric tensor categories

1. the category of all symmetric tensor categories, $\mathbf{SymTens}_{\mathbb{K}}$,
2. the category of all pre-Tannakian symmetric tensor categories, $\mathbf{PTann}_{\mathbb{K}}$,
3. the category of all pre-Tannakian symmetric tensor categories of moderate growth, $\mathbf{MdGr}_{\mathbb{K}}$.

We have the inclusions $\mathbf{MdGr}_{\mathbb{K}} \subset \mathbf{PTann}_{\mathbb{K}} \subset \mathbf{Tens}_{\mathbb{K}}$. Positive and negative classification results usually describe the structure of (one of) these categories.

Tannakian reconstruction

The following theorem shows that symmetric tensor functors between pre-Tannakian categories are quite special. In Corollary 4.3.9 we already showed that they have to be faithful, and the following theorem will show that they allow to reconstruct the domain category as representations in the target category.

Theorem 4.5.11 ([Del90, Théorème 8.17]). *Let $\mathcal{C}, \mathcal{D}$ be pre-Tannakian categories (where $\mathcal{D}$ contains at least one non-zero object), and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric tensor functor. Then there exists an affine group scheme G in $\mathcal{D}$ (a concept we will formally define in Chapter 6) such that the category of representations of G in $\mathcal{D}$ is equivalent to $\mathcal{C}$, i.e.*

$$\mathcal{C} \simeq \mathbf{Rep}_{\mathcal{D}}(G). \tag{4.39}$$

As a corollary, we observe that if a (2-)category of pre-Tannakian categories admits a weakly final object (i.e. an object to which every other object admits a morphism), then every object in this category is equivalent, as a symmetric tensor category, to the category of representations of some affine group scheme internal to the final category. The classification theorems discussed below can be interpreted as identifying such final objects in suitable (2-)categories of pre-Tannakian categories.

[5] The four names mentioned here are arguably among the most influential contributors. I would like to highlight, with some national pride, that two of them (Pierre Deligne and Kevin Coulembier) are Belgian!

Symmetric tensor categories of moderate growth

The classification of pre-Tannakian categories was started by Deligne in [Del02], with the following result.

Theorem 4.5.12 ([Del02, Proposition 0.5, Théorème 0.6]). *Let $\mathcal{C}$ be a pre-Tannakian category over an algebraically closed field of characteristic zero $\mathbb{K}$. There exists a symmetric tensor functor (unique up to isomorphism), called a* fibre functor*, to the category of finite-dimensional supervector spaces*

$$\mathcal{C} \to \mathbf{FinsVect}_{\mathbb{K}}. \tag{4.40}$$

Equivalently, there is an affine supergroup scheme G such that

$$\mathcal{C} \simeq \mathbf{Rep}G. \tag{4.41}$$

Generalising the above result 4.5.12 to fields $\mathbb{K}$ of positive characteristic $p > 0$ turned out to be very difficult. It was found that there exists a symmetric fusion category that does not admit a fibre functor to $\mathbf{FinsVect}_{\mathbb{K}}$, the *(universal) Verlinde category* Ver_p ([GK92; GM94; Ost20]), which contains $\mathbf{FinsVect}_{\mathbb{K}}$ as a fusion subcategory. This Verlinde category will be the subject of chapters 7 and 8.

More than a decade after Deligne's result, the following theorem by Ostrik brought marked a turning point in the structure theory in positive characteristic.

Theorem 4.5.13 ([Ost20, Theorem 1.5, Corollary 1.6]). *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Any symmetric fusion category $\mathcal{C}$ over $\mathbb{K}$ admits a symmetric tensor functor*

$$\mathcal{C} \to \mathrm{Ver}_p. \tag{4.42}$$

Equivalently, there exists an affine algebraic group G in Ver_p such that

$$\mathcal{C} \simeq \mathbf{Rep}_{\mathrm{Ver}_p}(G). \tag{4.43}$$

Later it was proved by Coulembier, Etingof, Ostrik (in [CEOK23, Theorem 1.1, Theorem 7.13]) that a a pre-Tannakian category admits a fibre functor to Ver_p if and only if it is of moderate growth and is *Frobenius exact* (based on earlier work [Cou20; EO21b]).

It was then realised that this Verlinde category is not sufficient for the full structure theory, and higher Verlinde categories Ver_{p^n} were introduced ([BE19; BEO23; Cou21]), with $\mathrm{Ver}_p \subseteq \mathrm{Ver}_{p^2} \subseteq \mathrm{Ver}_{p^3} \subseteq \dots$ After the construction of these categories, it was conjectured that they allow a generalisation of Deligne's result.

Conjecture 4.5.14 ([BEO23, Conjecture 1.4]). *Every pre-Tannakian category $\mathcal{C}$ over an algebraically closed field of characteristic $p > 0$ admits a symmetric tensor functor*

$$\mathcal{C} \to \mathrm{Ver}_{p^\infty}. \tag{4.44}$$

Equivalently, there is an affine group scheme G in Ver_{p^∞} such that

$$\mathcal{C} \simeq \mathbf{Rep}_{\mathrm{Ver}_{p^\infty}}(G). \tag{4.45}$$

This remains a conjecture until this day. However, Coulembier, Etingof, Ostrik have since proved a classification theorem that comes quite close.

Definition 4.5.15 (Injective tensor functors, [CEO24b, Lemma 3.1.1]). Let $\mathcal{C}, \mathcal{D}$ be pre-Tannakian categories. A symmetric tensor functor $F : \mathcal{C} \to \mathcal{D}$ is called *injective* if it is an equivalence between $\mathcal{C}$ and a (full) tensor subcategory of $\mathcal{D}$.

Remark 4.5.16. One can prove that for a symmetric tensor functor between pre-Tannakian categories, the following statements are equivalent ([CEO24b, Lemma 3.1.1])

1. F is injective,
2. F is fully faithful and sends simple objects to simple objects,
3. F is fully faithful and for every $A \in \mathrm{Ob}(\mathcal{C})$, every subobject of $F(A)$ is of the form $F(B)$ for some subobject B of A.

Definition 4.5.17 (Incompressible categories, [“cite [Definition˜3.2.1]–CEO24Incompressible˝]). A pre-Tannakian category is called *incompressible* if every braided tensor functor to any pre-Tannakian category is injective (i.e., every braided tensor functor out of this category is an embedding).

Theorem 4.5.18 ([CEO24b, Theorem 5.2.1]). *Every pre-Tannakian category of moderate growth, over an algebraically closed field of any characteristic, admits a symmetric tensor functor to some incompressible category of moderate growth. Equivalently, every pre-Tannakian category of moderate growth is equivalent to a representation category of some affine group scheme in an incompressible category of moderate growth.*

Deligne’s result 4.5.12 can then be captured in the following theorem.

Theorem 4.5.19. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The only incompressible categories of moderate growth are the category of finite-dimensional super-vector spaces, and the category of finite-dimensional vector spaces.*

Conjecture 4.5.14 then becomes.

Conjecture 4.5.20 ([BEO23, Conjecture 1.4] and [CEO24b, Conjecture B]). *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Every incompressible category of moderate growth over $\mathbb{K}$ is a tensor subcategory of Ver_{p^∞}.*

A generalisation of the result for Frobenius exact categories is still a conjecture, even for incompressible categories.

Conjecture 4.5.21 ([CEO24b, Conjecture C]). *A pre-Tannakian category over an algebraically closed field admits a symmetric tensor functor to an incompressible category if and only if it is of moderate growth.*

Symmetric tensor categories of non-moderate growth

We have now seen that pre-Tannakian categories of subexponential (i.e. moderate) growth can be classified as representation categories of affine group schemes. The following result shows that, even in characteristic zero, things go wrong when trying to give a classification theorem that is similar to the ones we have mentioned above for categories of non-moderate growth.

Theorem 4.5.22 ([CEO24b, Proposition 5.1.1]). *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Neither the category of all symmetric tensor categories (in which objects are allowed to have non-finite length) nor the category of all pre-Tannakian categories have a final object. As a consequence, there cannot exist one category $\mathcal{D}$ such that every pre-Tannakian or symmetric tensor category $\mathcal{C}$ is equivalent to a representation category of an affine group scheme over $\mathcal{D}$.*

So, something seems to go wrong for pre-Tannakian categories of superexponential (i.e. non-moderate) growth. For these categories, it is hard to even construct examples. In characteristic zero, Deligne and Milne constructed such categories ([DM82; Del07]).

Not a lot is known about these categories, but Andrew Snowden (among others) is doing some pioneering research in this direction using oligomorphic groups (see, for example, [HS22; HNS23; Sno23; Sno24]).

Part II

Semisimplification and Algebras in Symmetric Tensor Categories

5 Semisimplification

In this chapter, we introduce the main tool currently used to construct non-associative algebras in certain exotic symmetric tensor categories (see [Kan24; EEK25]): *semisimplification*. Our treatment of semisimplification will be quite general (more general than what I have seen in the existing literature). In particular, we will not restrict ourselves to symmetric or even pivotal tensor categories. As a result, our discussion applies to all braided tensor categories, including those that are neither symmetric nor balanced.

The guiding idea is simple: tensor categories can be viewed as categorified analogues of local rings, and semisimplification corresponds to forming the quotient by the maximal proper ideal. For braided tensor categories, this quotient corresponds to a categorification of the residue field of a commutative local ring.

Although the notion of semisimplification is not formally defined, outside the specific contexts we will encounter in this chapter, we attempt below to outline a very general procedure for semisimplification using some definitions that are not fully formal.

Definition 5.0.1 (Categorical ideals). Let $\mathbf{Struct}$ be a (2-)category whose objects are categories enriched over some monoidal category $\mathcal{M}$, and which is equipped with a collection BinOp of binary operations defined on all morphisms (including, for example, the composition). Let $\mathcal{C} \in \mathrm{Ob}\,(\mathbf{Struct})$ be such a structured category. An *ideal* $\mathcal{I} \leq \mathcal{C}$ is a collection of subobjects $\mathcal{I}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ (in $\mathcal{M}$) for all $A, B \in \mathrm{Ob}\,(\mathcal{C})$, such that

$$o(f, g), o(g, f) \in \mathcal{I} \text{ for all } f \in \mathcal{I}, g \in \mathrm{Hom}(\mathcal{C}), \text{ and } o \in \mathrm{BinOp}. \tag{5.1}$$

Generally (when $\mathcal{M}$ has a notion of quotients, e.g. when it is abelian), such ideals give rise to *quotients* of categories $\mathcal{C}/\mathcal{I} \in \mathrm{Ob}\,(\mathbf{Struct}')$, where $\mathbf{Struct} \subseteq \mathbf{Struct}'$ is some larger category. Such a quotient comes equipped with a canonical *quotient functor*, $\mathrm{quot}_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}/\mathcal{I} \in \mathrm{Hom}(\mathbf{Struct}')$.

We will see in Section 5.2.1 that this is a natural definition.

Definition 5.0.2 (Semisimplification). Let $\mathbf{Struct}, \mathbf{Struct}'$ be as in the above, and suppose further that $\mathbf{Struct}'$ is contained in the category of additive categories. Let $\mathcal{C} \in \mathrm{Ob}\,(\mathbf{Struct})$. If $\mathrm{rad}(\mathcal{C}) \leq \mathcal{C}$ is a minimal ideal among those $\mathcal{I} \leq \mathcal{C}$ for which $\mathcal{C}/\mathcal{I}$ is semisimple and Schur, or equivalently semisimple and abelian (in the sense that every such $\mathcal{I}$ contains $\mathrm{rad}(\mathcal{C})$), then the quotient $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ is called the *semisimplification* of $\mathcal{C}$.

Many of the results and ideas presented in this chapter are adapted from or inspired by [AKO02; EO21a].

5.1 Indecomposable and simple objects in abelian categories

Before discussing semisimplification in the settings of abelian and tensor categories, we begin with a more detailed study of indecomposable objects and their endomorphism rings in abelian categories. The motivation for this lies in our interest in ideals of morphisms in such categories. Indecomposable objects serve as the building blocks of abelian categories, and every morphism (in categories that admit decompositions into indecomposables) decomposes as a sum of morphisms between indecomposable objects. Consequently, understanding the structure of morphisms between indecomposables is an important step for gaining insight into the behaviour of general morphisms and ideals.

This section focuses specifically on endomorphisms of indecomposable objects. We will show that every such morphism is either an isomorphism or nilpotent. Furthermore, we will demonstrate that indecomposable objects can be recognised by their endomorphism rings.

This discussion is inspired by a similar treatment in the setting of modules found in [Ben84].

5.1.1 Some ring theory

In this small section, we review some background on local rings that will be used in this chapter.

We assume our rings to be unital, but not necessarily commutative.

Definition 5.1.1 (Jacobson radical). Let R be a ring. We define the *Jacobson radical* as

$$\mathrm{rad}(R) := \{x \in R \mid (\forall r \in R)(1_R - r \cdot x \text{ has a right inverse})\}. \tag{5.2}$$

It is not very hard to show that this is an ideal in R.

1. Let $x \in \mathrm{rad}(R)$, and let $r, y \in R$. Then $1_R - r \cdot y \cdot x$ has some right inverse, which shows that $y \cdot x \in \mathrm{rad}(R)$. This shows that $\mathrm{rad}(R)$ is closed under left multiplication.
2. Let $x, y \in \mathrm{rad}(R)$, and let $r \in R$. We know that $1_R - r \cdot x$ has some right inverse z, and we then find $1_R - r \cdot (x+y) = 1_R - r \cdot x - r \cdot y = (1_R - r \cdot x) \cdot (1_R - z \cdot r \cdot y)$. As both terms have right inverses, we find that $x + y \in \mathrm{rad}(R)$. It is also clear that $\mathrm{rad}(R)$ is closed under taking additive inverses, which shows that $\mathrm{rad}(R)$ is a subgroup of $(R, +)$.
3. Let $x \in \mathrm{rad}(R)$, and let $r, y \in R$. We know that $1_R - y \cdot r \cdot x$ has some right inverse z, which implies that $(1_R - r \cdot x \cdot y) \cdot (1_R + r \cdot x \cdot z \cdot y) = 1_R - r \cdot x \cdot y + r \cdot x \cdot (1_R - y \cdot r \cdot x) \cdot z \cdot y = 1_R$. This shows that $\mathrm{rad}(R)$ is closed under right multiplication.

We could have equivalently defined the Jacobson radical as

$$\mathrm{rad}(R) = \{x \in R \mid (\forall r \in R)(1_R - r \cdot x \text{ is invertible})\}. \tag{5.3}$$

For $x \in \mathrm{rad}(R)$ and $r \in R$, we know that $1_R - r \cdot x$ has a right inverse y. We thus find $y = 1_R + r \cdot x \cdot y$, which again has a right inverse z through (3). Now, $z = 1_R \cdot z = (1_R - r \cdot x) \cdot y \cdot z = 1_R - r \cdot x$, which shows that $1_R - r \cdot x$ is invertible with inverse y.

Similarly, we can prove an equality of ideals

$$\{x \in R \mid (\forall r \in R)(1_R - x \cdot r \text{ has a left inverse})\} = \{x \in R \mid (\forall r \in R)(1_R - x \cdot r \text{ is invertible})\}. \tag{5.4}$$

Finally, we have

$$\{x \in R \mid (\forall r \in R)(1_R - x \cdot r \text{ is invertible})\} = \{x \in R \mid (\forall r \in R)(1_R - r \cdot x \text{ is invertible})\}. \tag{5.5}$$

This follows from the fact that both these sets are now known to be ideals in R, which implies that they coincide with $\{x \in R \mid (\forall r, s \in R)(1_R - r \cdot x \cdot s \text{ is invertible})\}$.

We summarise with the following equivalent characterisations of the Jacobson radical

$$\begin{aligned} \mathrm{rad}(R) &= \{x \in R \mid (\forall r \in R)(1_R - r \cdot x \text{ has a right inverse})\} && (5.6)\\ &= \{x \in R \mid (\forall r \in R)(1_R - r \cdot x \text{ is invertible})\} && (5.7)\\ &= \{x \in R \mid (\forall r \in R)(1_R - x \cdot r \text{ has a left inverse})\} && (5.8)\\ &= \{x \in R \mid (\forall r \in R)(1_R - x \cdot r \text{ is invertible})\}. && (5.9) \end{aligned}$$

Proposition 5.1.2. *Let R be a ring. The following properties are equivalent*

1. *R has a unique maximal proper left ideal, and it coincides with the Jacobson radical,*
2. *R has a unique maximal proper right ideal, and it coincides with the Jacobson radical,*
3. *the sum of any two non-invertible elements of R is once again non-invertible,*
4. *for all $x \in R$, either x or $1_R - x$ is invertible.*

If one of these properties is satisfied, then the Jacobson radical (and thus the unique maximal proper left or right ideals) coincides with the set of non-invertible.

Proof. It is immediately clear that (3) implies (4). This also shows that $I = \{x \in R \mid x \text{ is not invertible}\}$ is an ideal of R. (3) shows that it is closed under addition, and (4) shows that it is closed under left and right multiplication: suppose that $x \in I, r \in R$ are such that $r \cdot x$ (resp. $x \cdot r$) is invertible

1. if r is invertible, $x = r^{-1} \cdot r \cdot x$ is a product of invertible elements, hence invertible,
2. if r is not invertible, then $1_R - r$ is invertible, which means that $r \cdot x - x$ is not invertible by the previous, and this then implies that $r \cdot x = r \cdot x - x + x$ is not invertible by (3).

It is clear that this ideal is the unique maximal proper left and right ideal, and that it coincides with the Jacobson radical through (4). We conclude that (3) implies (1) and (2).

To show that (4) implies (3), suppose that x, y are non-invertible such that $x + y$ is invertible with inverse z. As in the above, we find that $x \cdot z$ and $y \cdot z$ are not invertible. This implies that $x \cdot z = 1_R - y \cdot z$ is invertible, which is a contradiction.

To show that (1) implies (4), suppose that there exists x such that both x and $1_R - x$ are not invertible.

If x has a left inverse r, then $(x \cdot r - 1_R) \cdot x = 0$. This implies that $x \cdot r - 1_R$ is contained in the proper left ideal $\mathrm{Ker}(- \cdot x)$. Because the Jacobson radical is the maximal proper left ideal, this implies that $x \cdot r - 1_R \in \mathrm{rad}(R)$. In particular, this implies that $x \cdot r$ is invertible, and thus that x has a right inverse. We conclude that x is invertible. Similarly, we show that $1_R - x$ has a left inverse if and only if it is invertible.

So, we may assume that the left ideals generated by x and $1_R - x$ are proper. This implies that these ideals are both contained in the unique maximal left ideal, and thus that $x + 1_R - x = 1_R$ is contained in this ideal. This is clearly a contradiction.

The proof that (2) implies (4) is similar. ■

Definition 5.1.3 (Local rings). Let R be a ring. R is called *local* if it satisfies one of the equivalent conditions in Proposition 5.1.2.

Proposition 5.1.4. *Let R be a local ring, and let $I \leq R$ be an ideal in R. R/I is a local ring.*

Proof. R has a unique maximal proper left ideal M. Let $\mathrm{quot} : R \to R/I$ be the projection of R onto R/I. Let N be any maximal proper left ideal in R/I. $\mathrm{quot}^{-1}(N)$ is then a proper left ideal in R, and we thus find $\mathrm{quot}^{-1}(N) \subseteq M$. Suppose now that $\mathrm{quot}^{-1}(N) \neq M$. This implies that $M/I = R/I$, and thus that $M = R$ as this implies that $1_R \in M$. We conclude that $N = \mathrm{quot}(M)$, and thus that $\mathrm{quot}(M)$ is the unique maximal proper left ideal in R/I. ■

5.1.2 Some technical results on endomorphisms under additive isomorphisms

In this chapter, we will require results showing that, in two distinct settings, any morphism between indecomposable objects is either an isomorphism or nilpotent. With the benefit of hindsight, we have opted to present a more general (if slightly more technical) treatment that captures both situations at once.

Definition 5.1.5 (Nilpotent morphisms in abelian categories equipped with an isomorphism). Let $\mathcal{C}$ be an abelian category, and let $F : \mathcal{C} \to \mathcal{C}$ be an additive isomorphism, by which we mean that F is an additive functor and there exists an inverse functor $F^{-1} : \mathcal{C} \to \mathcal{C}$ such that $F \circ F^{-1} = F^{-1} \circ F = \mathrm{id}_{\mathcal{C}}$.

For any object $A \in \mathrm{Ob}(\mathcal{C})$, any morphism $f : A \to F(A)$, and any $n \geq 0$, we define

$$f^{(n)} := F^{n-1}(F) \circ F^{n-2}(f) \circ \cdots \circ F(f) \circ f : A \to F^n(A). \tag{5.10}$$

f is called *nilpotent* if there exists $n \geq 0$ such that $f^{(n)} = 0$.

Remark 5.1.6. An additive isomorphism $F : \mathcal{C} \to \mathcal{C}$ gives two adjoint pairs, (F, F^{-1}) and (F^{-1}, F), which shows that both F and F^{-1} preserve all limits and colimits. In particular, they are exact.

The following lemma and its proof are adapted from the fitting lemma for modules [Ben84, Lemma 1.3.2], generalised to our categorical setting. Although the proof is inspired by a relatively simple argument in the setting of modules, its generalisation is not entirely straightforward.

Lemma 5.1.7. *Let $\mathcal{C}$ be an abelian category equipped with an additive isomorphism $F : \mathcal{C} \to \mathcal{C}$, let $A \in \mathrm{Ob}(\mathcal{C})$ be an object of finite length in $\mathcal{C}$ (note that $F(A)$ is then of the same length by applying F to a Jordan-Hölder filtration), and let $f : A \to F(A)$ be a morphism. There exists N such that for all $n \geq N$ the short exact sequences*

$$\begin{array}{c} 0 \longrightarrow \mathrm{Ker}(f^{(n)}) \xrightarrow{\ \ker(f^{(n)})\ } A \xrightarrow{\ \mathrm{coim}(f^{(n)})\ } \mathrm{Coim}(f^{(n)}) \longrightarrow 0 \\ \\ 0 \longrightarrow \mathrm{Im}(f^{(n)}) \xrightarrow{\ \mathrm{im}(f^{(n)})\ } F^n(A) \xrightarrow{\ \mathrm{coker}(f^{(n)})\ } \mathrm{Coker}(f^{(n)}) \longrightarrow 0 \end{array} \tag{5.11}$$

are split. This implies that

$$A = \mathrm{Ker}(f^{(n)}) \oplus \mathrm{Coim}(f^{(n)}) \text{ and } F^n(A) = \mathrm{Coker}(f^{(n)}) \oplus \mathrm{Im}(f^{(n)}). \tag{5.12}$$

Proof. Let $n \geq 0$ be arbitrary. As $F^{-n}(f^{(2n)}) = f^{(n)} \circ F^{-n}(f^{(n)})$ and $F^n(f^{(n)}) \circ f^{(n)} = f^{(2n)}$, we find

$$\mathrm{coker}(F^{-n}(f^{(2n)})) \circ (f^{(n)} \circ \mathrm{im}(F^{-n}(f^{(n)}))) = 0 \text{ and } \mathrm{coim}(F^n(f^{(n)})) \circ f^{(n)} \circ \ker(f^{(2n)}) = 0. \tag{5.13}$$

This implies that we have uniquely induced morphisms α_n, β_n such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Coim}(f^{(2n)}) & \overset{\exists!\beta_n}{\dashrightarrow} & \mathrm{Coim}(F^n(f^{(n)})) \\ {\scriptstyle \mathrm{coim}(f^{(2n)})}\uparrow & & \uparrow{\scriptstyle \mathrm{coim}(F^n(f^{(n)}))} \\ A & \xrightarrow{\ f^{(n)}\ } & F^n(A) \\ {\scriptstyle \mathrm{im}(F^{-n}(f^{(n)}))}\uparrow & & \uparrow{\scriptstyle \mathrm{im}(F^{-n}(f^{(2n)}))} \\ \mathrm{Im}(F^{-n}(f^{(n)})) & \overset{\exists!\alpha_n}{\dashrightarrow} & \mathrm{Im}(F^{-n}(f^{(2n)})) \end{array} . \tag{5.14}$$

We claim that α_n is an epimorphism for all n, and that β_n is a monomorphism for all n. For this, note that

$$F^{-n}(f^{(2n)}) = \mathrm{im}(F^{-n}(f^{(2n)})) \circ \alpha_n \circ \mathrm{coim}(F^{-n}(f^{(n)})), \text{ and} \tag{5.15}$$
$$F^n(f^{(n)}) = \mathrm{im}(F^n(f^{(n)})) \circ \beta_n \circ \mathrm{coim}(f^{(2n)}), \tag{5.16}$$

which implies that $\alpha_n \circ \mathrm{coim}(F^{-n}(f^{(n)})) = \mathrm{coim}(F^{-n}(f^{(2n)}))$ is an epimorphism, and that $\mathrm{im}(F^n(f^{(n)})) \circ \beta_n = \mathrm{im}(F^n(f^{(n)}))$ is a monomorphism. This implies in turn that α_n is an epimorpism, and that β_n is a monomorphism.

We claim that, in addition, α_n is a monomorphism, and β_n an epimorphism, for n large enough.

Let $m \geq n$.

1. $f^{(m)} \circ \ker(f^{(n)}) = 0$ implies that there exists a unique morphism $k_{n,m} : \mathrm{Ker}(f^{(n)}) \to \mathrm{Ker}(f^{(m)})$ such that $\ker(f^{(m)}) \circ k_{n,m} = \ker(f^{(n)})$.

2. $f^{(m)} \circ \ker(f^{(n)}) = 0$ also implies that $\mathrm{coim}(f^{(m)}) \circ \ker(f^{(n)}) = 0$, and thus that there exists a unique morphism $c_{n,m} : \mathrm{Coim}(f^{(n)}) \to \mathrm{Coim}(f^{(m)})$ such that $c_{n,m} \circ \mathrm{coim}(f^{(n)}) = \mathrm{coim}(f^{(m)})$.

3. $\mathrm{coker}(f^{(n)}) \circ F^{n-m}(f^{(m)}) = 0$ implies that $\mathrm{coker}(f^{(n)}) \circ \mathrm{im}(F^{n-m}(f^{(m)})) = 0$, and thus that there exists a unique morphism $i_{n,m} : \mathrm{Im}(F^{n-m}(f^{(m)})) \to \mathrm{Im}(f^{(n)})$ such that $\mathrm{im}(f^{(n)}) \circ i_{n,m} = \mathrm{im}(F^{n-m}(f^{(m)}))$.

As A is of finite length, we know that there exists N such that for all $m \geq n \geq N$, $i_{n,m}, k_{n,m}, c_{n,m}$ are isomorphisms. For such n, we then find

$$F^{-n}(f^{(2n)}) = \mathrm{im}(F^{-n}(f^{(2n)})) \circ \mathrm{coim}(F^{-n}(f^{(2n)})), \text{ and} \tag{5.17}$$

$$\begin{aligned} F^{-n}(f^{(2n)}) &= f^{(n)} \circ F^{-n}(f^{(n)}) = \mathrm{im}(f^{(n)}) \circ \mathrm{coim}(f^{(n)}) \circ \mathrm{im}(F^{-n}(f^{(n)})) \circ \mathrm{coim}(F^{n}(f^{(n)})) \\ &= \mathrm{im}(F^{-n}(f^{(2n)})) \circ i_{n,2n}^{-1} \circ \mathrm{coim}(f^{(n)}) \circ \mathrm{im}(F^{-n}(f^{(n)})) \circ F^{-n}(c_{n,2n}^{-1}) \circ \mathrm{coim}(F^{-n}(f^{(2n)}))' \end{aligned} \tag{5.18}$$

which implies that $\mathrm{coim}(f^{(n)}) \circ \mathrm{im}(F^{-n}(f^{(n)}))$ is an isomorphism.

Using

$$\mathrm{im}(F^{-n}(f^{(2n)})) \circ \alpha_n = f^{(n)} \circ \mathrm{im}(F^{-n}(f^{(n)})) = \mathrm{im}(f^{(n)}) \circ \mathrm{coim}(f^{(n)}) \circ \mathrm{im}(F^{-n}(f^{(n)})), \text{ and} \tag{5.19}$$

$$\beta_n \circ \mathrm{coim}(f^{(2n)}) = \mathrm{coim}(F^{n}(f^{(n)})) \circ f^{(n)} = \mathrm{coim}(F^{n}(f^{(n)})) \circ \mathrm{im}(f^{(n)}) \circ \mathrm{coim}(f^{(n)}), \tag{5.20}$$

we find that α_n is a monomorphism (and thus an isomorphism), and that β_n is an epimorphism (and thus an isomorphism) for $n \geq N$.

Let us now define, for $n \geq N$

$$\gamma_n := \mathrm{id}_A - \mathrm{im}(F^{-n}(f^{(n)})) \circ \alpha_n^{-1} \circ i_{n,2n}^{-1} \circ \mathrm{coim}(f^{(n)}), \text{ and} \tag{5.21}$$

$$\zeta_n := \mathrm{id}_{F^n(A)} - \mathrm{im}(f^{(n)}) \circ c_{n,2n}^{-1} \circ \beta_n^{-1} \circ \mathrm{coim}(F^{n}(f^{(n)})). \tag{5.22}$$

We obtain

$$\left.\begin{aligned} f^{(n)} \circ \gamma_n &= f^{(n)} - f^{(n)} \circ \mathrm{im}(F^{-n}(f^{(n)})) \circ \alpha_n^{-1} \circ i_{n,2n}^{-1} \circ \mathrm{coim}(f^{(n)}) \\ &= f^{(n)} - \mathrm{im}(F^{-n}(f^{(2n)})) \circ i_{n,2n}^{-1} \circ \mathrm{coim}(f^{(n)}) \\ &= f^{(n)} - \mathrm{im}(f^{(n)}) \circ \mathrm{coim}(f^{(n)}) \\ &= 0 \end{aligned}\right., \text{ and} \tag{5.23}$$

$$\left.\begin{aligned} \zeta_n \circ f^{(n)} &= f^{(n)} - \mathrm{im}(f^{(n)}) \circ c_{n,2n}^{-1} \circ \beta_n^{-1} \circ \mathrm{coim}(F^{n}(f^{(n)})) \circ f^{(n)} \\ &= f^{(n)} - \mathrm{im}(f^{(n)}) \circ c_{n,2n}^{-1} \circ \mathrm{coim}(f^{(2n)}) \\ &= f^{(n)} - \mathrm{im}(f^{(n)}) \circ \mathrm{coim}(f^{(n)}) \\ &= 0 \end{aligned}\right.. \tag{5.24}$$

This implies that there exist uniquely induced morphisms $s : A \to \mathrm{Ker}(f^{(n)})$ and $t : \mathrm{Coker}(f^{(n)}) \to F^n(A)$ such that $\ker(f^{(n)}) \circ s = \gamma_n$ and $t \circ \mathrm{coker}(f^{(n)}) = \zeta_n$.

As $\ker(f^{(n)}) \circ s \circ \ker(f^{(n)}) = \gamma_n \circ \ker(f^{(n)}) = \ker(f^{(n)})$ and $\mathrm{coker}(f^{(n)}) \circ t \circ \mathrm{coker}(f^{(n)}) = \mathrm{coker}(f^{(n)}) \circ \zeta_n = \mathrm{coker}(f^{(n)})$, we find $s \circ \ker(f^{(n)}) = \mathrm{id}_{\mathrm{Ker}(f^{(n)})}$ and $\mathrm{coker}(f^{(n)}) \circ t = \mathrm{id}_{\mathrm{Coker}(f^{(n)})}$. This shows that $\ker(f^{(n)})$ is a split monomorphism, and that $\mathrm{coker}(f^{(n)})$ is a split epimorphism. We thus obtain the split short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathrm{Ker}(f^{(n)}) \xrightarrow{\ker(f^{(n)})} A \xrightarrow{\mathrm{coim}(f^{(n)})} \mathrm{Coim}(f^{(n)}) \longrightarrow 0 \\ 0 \longrightarrow \mathrm{Im}(f^{(n)}) \xrightarrow{\mathrm{im}(f^{(n)})} F^n(A) \xrightarrow{\mathrm{coker}(f^{(n)})} \mathrm{Coker}(f^{(n)}) \longrightarrow 0 \end{gathered}. \tag{5.25}$$

■

Since the result above provides a direct sum decomposition for arbitrary objects, its application to indecomposable objects yields an interesting consequence.

Proposition 5.1.8. *Let $\mathcal{C}$ be an abelian category equipped with an additive isomorphism $F : \mathcal{C} \to \mathcal{C}$, let $A \in \mathrm{Ob}(\mathcal{C})$ be an indecomposable object of finite length in $\mathcal{C}$ (note that $F^n(A)$ is then indecomposable too), and let $f : A \to F(A)$ be a morphism. Either f is an isomorphism, or f is nilpotent.*

Proof. Lemma 5.1.7 gives two split short exact sequences

$$\begin{array}{c} 0 \longrightarrow \operatorname{Ker}(f^{(n)}) \xrightarrow{\ker(f^{(n)})} A \xrightarrow{\operatorname{coim}(f^{(n)})} \operatorname{Coim}(f^{(n)}) \longrightarrow 0 \\ . \\ 0 \longrightarrow \operatorname{Im}(f^{(n)}) \xrightarrow{\operatorname{im}(f^{(n)})} F^n(A) \xrightarrow{\operatorname{coker}(f^{(n)})} \operatorname{Coker}(f^{(n)}) \longrightarrow 0 \end{array} \tag{5.26}$$

As A is indecomposable, we know that either $\ker(f^{(n)}) = 0$ or $\operatorname{coim}(f^n) = 0$.

Suppose that $\ker(f^{(n)}) = 0$, then we know that $\operatorname{coim}(f^{(n)})$ is an isomorphism, and thus that $f^{(n)}$ is a split monomorphism as $\operatorname{im}(f^{(n)})$ is a split monomorphism. However, as $F^n(A)$ is indecomposable, this implies that $\operatorname{im}(f^{(n)})$ is an isomorphism, hence that $f^{(n)}$ is an isomorphism. This implies that f is a split monomorphism, and after applying F^{-n} this also implies that f is a split epimorphism. We conclude that f is an isomorphism.

Suppose that $\operatorname{coim}(f^{(n)}) = 0$, then trivially $f^{(n)} = 0$ as $f^{(n)} = \operatorname{im}(f^{(n)}) \circ \operatorname{coim}(f^{(n)})$. ■

5.1.3 Recognising indecomposable objects by their endomorphism rings

Applying the above discussion to $F = \mathrm{id}_{\mathcal{C}}$ results in the following statement.

Proposition 5.1.9. *Let $\mathcal{C}$ be an abelian category, let $A \in \operatorname{Ob}(\mathcal{C})$ be an indecomposable object of finite length, and let $f : A \to A$ be an endomorphism. Then either f is an isomorphism, or f is nilpotent.*

As $F = \mathrm{id}_{\mathcal{C}}$, Proposition 5.1.9 is a statement about the structure of a ring. The next result, which is a straightforward generalisation of [Ben84, Lemma 1.3.3], shows that this ring is local.

Corollary 5.1.10. *Let $\mathcal{C}$ be an abelian category, and let $A \in \operatorname{Ob}(\mathcal{C})$ be an indecomposable object of finite length. $\operatorname{Hom}_{\mathcal{C}}(A, A)$ is a local ring when equipped with the ring structure induced by addition and composition. The unique maximal left and right ideal in this ring consists of all nilpotent morphisms.*

Proof. Suppose that I is a maximal proper left ideal in $\operatorname{Hom}_{\mathcal{C}}(A, A)$, and let f be any morphism not in I. We will prove that f is an isomorphism, which shows that I consists of all non-invertible morphisms.

As $f \notin I$, and I is maximal, we conclude that there exist $g \in \operatorname{Hom}_{\mathcal{C}}(A, A)\,, h \in I$ such that $g \circ f + h = \mathrm{id}_A$. Proposition 5.1.9 shows that $h^n = 0$ for some n. We then find $(\mathrm{id}_A + h + \cdots + h^{n-1}) \circ g \circ f = (\mathrm{id}_A + h + \cdots + h^{n-1}) \circ (\mathrm{id}_A - h) = \mathrm{id}_A$, which shows that f is a split monomorphism. As a split monomorphism into an indecomposable object, or equivalently because f is not nilpotent as a monomorphism, we conclude that f is an isomorphism. ■

As with Schur's lemma over algebraically closed fields (see 2.4.10), this statement becomes somewhat stronger when the category is enriched over an algebraically closed field.

Corollary 5.1.11. *Let $\mathcal{C}$ be an abelian category that is $\mathbf{Vect}_{\mathbb{K}}$-enriched with $\mathbb{K}$ an algebraically closed field, and let $A \in \operatorname{Ob}(\mathcal{C})$ be an indecomposable object of finite length. We have $\operatorname{Hom}_{\mathcal{C}}(A, A)\,/\mathrm{nilpotents} \cong \mathbb{K}$ as algebras, which shows that any endomorphism $f : A \to A$ can be written as $f = \lambda \mathrm{id}_A + \mathrm{nilpotent}$ for some $\lambda \in \mathbb{K}$.*

Proof. This follows from Corollary 5.1.9 and the fact that division algebras over algebraically closed fields are isomorphic to that field. ■

It turns out that the converse of the above statement also holds: objects of finite length whose endomorphism rings are local are indecomposable. Moreover, this remains true in a more general setting.

Proposition 5.1.12. *Let $\mathcal{C}$ be a pre-additive category, and let $A \in \operatorname{Ob}(\mathcal{C})$. If $\operatorname{Hom}_{\mathcal{C}}(A, A)$ is a local ring, then A is indecomposable.*

Proof. Suppose that this is not true, and thus that there exist non-zero objects $B, C \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\mathrm{inc}_B : B \to A, \mathrm{inc}_C : C \to A, \mathrm{proj}_B : A \to B, \mathrm{proj}_C : A \to C$ such that $\mathrm{inc}_B \circ \mathrm{proj}_B + \mathrm{inc}_C \circ \mathrm{proj}_C = \mathrm{id}_A, \mathrm{proj}_B \circ \mathrm{inc}_B = \mathrm{id}_B, \mathrm{proj}_C \circ \mathrm{inc}_C = \mathrm{id}_C$. Note that $\mathrm{inc}_B \circ \mathrm{proj}_B, \mathrm{inc}_C \circ \mathrm{proj}_C$ are not isomorphisms, as this would imply that $\mathrm{inc}_B, \mathrm{inc}_C$ are isomorphisms, and thus that $A \cong B, C$ (which is not possible as this would imply that $\mathrm{inc}_C \circ \mathrm{proj}_C, \mathrm{inc}_B \circ \mathrm{proj}_B = 0$, and thus $\mathrm{inc}_C, \mathrm{inc}_B = 0$, but these are monomorphisms from non-zero objects). We conclude that $\mathrm{inc}_B \circ \mathrm{proj}_B, \mathrm{inc}_C \circ \mathrm{proj}_C$ are elements of the unique maximal left and right ideal of $\mathrm{Hom}_{\mathcal{C}}(A, A)$. This is a contradiction as this would imply that $\mathrm{inc}_B \circ \mathrm{proj}_B + \mathrm{inc}_C \circ \mathrm{proj}_C = \mathrm{id}_A$ is contained in a proper ideal. ∎

Corollary 5.1.13. *Let $\mathcal{C}$ be an abelian category. An object $A \in \mathrm{Ob}(\mathcal{C})$ of finite length is indecomposable if and only if $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a local ring.*

Proof. This follows from Corollary 5.1.10 and Proposition 5.1.12. ∎

5.1.4 Recognising simple objects by their endomorphism rings

Schur's lemma 2.4.9 shows that the endomorphism ring of a simple object is a division ring, and one could be tempted by the above results to try and prove the converse. However, it turns out that the converse to Schur's lemma is not necessarily true!

Remark 5.1.14. Even in very well-behaved settings, the converse of Schur's lemma may fail. For instance, [EGNO15, Example 4.3.12] describes a ring category where the monoidal unit is not simple, yet its endomorphism ring is a field, and hence a division ring.

However, we can prove the following result, which shows that the converse of Schur's lemma holds when morphisms $A \to B$ and $B \to A$ are "paired".

Proposition 5.1.15. *Let $\mathcal{C}$ be a pre-additive category (e.g. abelian), and let $A \in \mathrm{Ob}(\mathcal{C})$ be an artinian object such that for any simple object $B \in \mathrm{Ob}(\mathcal{C})$, we have $\mathrm{Hom}_{\mathcal{C}}(A, B) \neq 0$ whenever $\mathrm{Hom}_{\mathcal{C}}(B, A) \neq 0$. If $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a division ring, then A is a simple object.*

Proof. As A is artinian, we know that it has a simple subobject (B, i). Suppose that $i \neq 0$, then we find $\mathrm{Hom}_{\mathcal{C}}(A, B) \neq 0$ through the assumption on A. Let $p \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be non-zero. Note that the fact that i is a monomorphism then implies that $i \circ p \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ is non-zero. As a non-zero morphism, $i \circ p$ is an isomorphism by assumption. We can thus conclude that i is a split epimorphism, and since it is also a monomorphism, we see that i is an isomorphism. We conclude that A has no non-zero proper subobjects, and thus that A is simple. ∎

One class of pre-additive categories that exhibits such a pairing between the hom-spaces $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(B, A)$ is the class of semisimple Schur categories.

Corollary 5.1.16. *Let $\mathcal{C}$ be a semisimple Schur category. An object $A \in \mathrm{Ob}(\mathcal{C})$ is simple if and only if its endomorphism ring $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a division ring.*

Proof. Let $A \in \mathrm{Ob}(\mathcal{C})$ be an arbitrary object, and let $B \in \mathrm{Ob}(\mathcal{C})$ be a simple object. If $A \cong A_1 \oplus \cdots \oplus A_n$ is a decomposition into simple objects, then $\mathrm{Hom}_{\mathcal{C}}(B, A) \neq 0$ if and only if $A_i \cong B$ for some $i \in \{1, \ldots, n\}$. It is immediately clear that then also $\mathrm{Hom}_{\mathcal{C}}(A, B) \neq 0$. ∎

5.2 Ideals in categories enriched over commutative rings

5.2.1 An oidification of ideals in algebras

In Chapter 1, we discussed groupoids and how they are the oidification (or horizontal categorification) of groups. We can carry out a similar process for algebras over a commutative ring R, which leads to the notion of R-*algebroids* (or *ringoids*[1] when $R = \mathbb{Z}$). It is clear that any R-algebra defines a ${}_R\mathbf{Mod}$-enriched category with a single object. This shows that R-algebroids are precisely ${}_R\mathbf{Mod}$-enriched categories.

As an oidification of algebras, it is natural to expect a corresponding notion of ideals in ${}_R\mathbf{Mod}$-enriched categories. Since the algebroid structure lives on the morphisms, such ideals should be defined in terms of the hom-sets.

Let $\mathcal{C}$ be the category with a single object $\star$, corresponding to some R-algebra A. An ideal in A corresponds to a subgroup $(I, +) \leq (\mathrm{Hom}_{\mathcal{C}}(\star, \star), +)$ that is closed under left or right composition (i.e., multiplication) with arbitrary morphisms.

This motivates the following natural notion of ideals in pre-additive categories.

Definition 5.2.1 (Ideals in categories enriched over commutative rings, [AKO02, § 1.3]). Let R be a commutative ring, and let $\mathcal{C}$ be a ${}_R\mathbf{Mod}$-enriched category. An *ideal* $\mathcal{I}$ in $\mathcal{C}$ consists of a collection $(\mathcal{I}(A, B), +) \leq (\mathrm{Hom}_{\mathcal{C}}(A, B), +)$ of subgroups for all $A, B \in \mathrm{Ob}(\mathcal{C})$, such that for all $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathcal{I}(B, C), g \in \mathrm{Hom}_{\mathcal{C}}(A, B), h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$

$$f \circ g \in \mathcal{I}(A, C) \text{ and } h \circ f \in \mathcal{I}(B, D). \tag{5.27}$$

If only the first of those two properties holds, then $\mathcal{I}$ is called a *right ideal*, and if only the second of those two property holds, then $\mathcal{I}$ is called a *left ideal*.

We write $\mathcal{I} \leq \mathcal{C}$ to indicate that $\mathcal{I}$ is an ideal in $\mathcal{C}$.

Ideals in ${}_R\mathbf{Mod}$-enriched categories are instances of Definition 5.0.1, with $\mathbf{Struct} = {}_R\mathbf{Cat}$ consisting of ${}_R\mathbf{Mod}$-enriched categories, $\mathcal{M} = {}_R\mathbf{Mod}$, and $\mathrm{BinOp} = \{\circ\}$.

Remark 5.2.2. We can make the oidification even more explicit by introducing a groupoid corresponding to the abelian group $(A, +)$ in the algebraic setting. To this end, we define a ${}_R\mathbf{Mod}$-enriched groupoid $(\mathrm{Hom}(\mathcal{C}), +)$, where

1. $\mathrm{Ob}\big(\mathrm{Hom}(\mathcal{C})\big) =$ the hom-sets in $\mathcal{C}$,
2. $\mathrm{Hom}_{\mathrm{Hom}(\mathcal{C})}\big(\mathrm{Hom}_{\mathcal{C}}(A, B), \mathrm{Hom}_{\mathcal{C}}(C, D)\big) = \varnothing$ if $A \neq C$ or $B \neq D$, and $\mathrm{Hom}_{\mathcal{C}}(A, B)$ otherwise,
3. for $f, g \in \mathrm{Hom}_{\mathrm{Hom}(\mathcal{C})}\big(\mathrm{Hom}_{\mathcal{C}}(A, B), \mathrm{Hom}_{\mathcal{C}}(A, B)\big) = \mathrm{Hom}_{\mathcal{C}}(A, B)$, we define composition by $f \circ g := f + g$.

The oidification of a subgroup $(I, +) \leq (A, +)$ is then a subgroupoid $(\mathcal{I}, +) \subseteq (\mathrm{Hom}(\mathcal{C}), +)$, which defines subgroups $(\mathcal{I}(A, B), +) \leq (\mathrm{Hom}_{\mathcal{C}}(A, B), +)$ for all[2] $A, B \in \mathrm{Ob}(\mathcal{C})$, such that $g \circ \mathcal{I} \subseteq \mathcal{I}$ (respectively, $\mathcal{I} \circ g \subseteq \mathcal{I}$) for all morphisms g, where $g \circ \mathcal{I} := \{g \circ f \mid f \in \mathcal{I} \text{ such that } g \circ f \text{ is defined}\}$. Note that this condition implies that $\mathcal{I}(A, B) \leq \mathrm{Hom}_{\mathcal{C}}(A, B)$ as R-modules.

Just as ideals in ring theory allow the construction of quotient rings, ideals in categories allow us to define quotients of categories.

Definition 5.2.3 (Quotient of categories enriched over commutative rings, [AKO02, § 1.3]). Let R be a commutative ring, let $\mathcal{C}$ be a ${}_R\mathbf{Mod}$-enriched category, and let $\mathcal{I}$ be an ideal in $\mathcal{C}$. We can then define a new ${}_R\mathbf{Mod}$-enriched category $\mathcal{C}/\mathcal{I}$, called the *quotient of* $\mathcal{C}$ *by* $\mathcal{I}$, with the following data

[1] *Ringoids* by John Baez, [Bae06], is a nice blog post about ringoids and enriched category theory.

[2] One could argue that for a proper oidification, one would only need to specify $\mathcal{I}(A, B)$ for some $A, B \in \mathrm{Ob}(\mathcal{C})$. However, can always set $\mathcal{I}(A, B) = 0$ for all other objects. This does not cause issues in the definition of ideals, as $g \circ 0 = 0$ follows from $g \circ 0 + g \circ 0 = g \circ 0$.

1. $\mathrm{Ob}\left(\mathcal{C}/\mathcal{I}\right) := \mathrm{Ob}\left(\mathcal{C}\right)$,
2. for $A, B \in \mathrm{Ob}\left(\mathcal{C}/\mathcal{I}\right)$, we set $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,B) := \mathrm{Hom}_{\mathcal{C}}(A,B)\,/\mathcal{I}(A,B)$,
3. the composition is inherited from $\mathcal{C}$ (and is well-defined through the left and right composition properties of ideals).

We obtain a canonical (R-linear) quotient functor

$$\mathrm{quot}_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}/\mathcal{I} : A \mapsto A \text{ and } (f : A \to B) \mapsto (f + \mathcal{I}(A,B) : A \to B). \tag{5.28}$$

Remark 5.2.4. This construction shows that, in Definition 5.0.1, $\mathbf{Struct}' = {}_R\mathbf{Cat}' = {}_R\mathbf{Cat} = \mathbf{Struct}$ in the ${}_R\mathbf{Mod}$-enriched setting. This equality $\mathbf{Struct} = \mathbf{Struct}'$ does not hold generally.

In what follows we will usually be interested in a structure category $\mathbf{Struct}$ that is strictly included in ${}_R\mathbf{Cat}$, for example: Schur categories, (artinian) abelian categories, (symmetric) tensor categories, …

In these settings, $\mathbf{Struct}$ and $\mathbf{Struct}'$ will usually not be equal. For example; quotients of abelian categories need not be abelian.

In the following sections, we will often restrict ourselves to the case $R = \mathbb{Z}$. This is the most general setting: any ${}_R\mathbf{Mod}$-enriched category is pre-additive, and any R-linear ideal is also an ideal in the pre-additive sense.

5.2.2 Ideals and indecomposable objects

Next, we want to show that, as with any other structure, understanding ideals on indecomposable objects suffices to determine them fully.

Proposition 5.2.5 ([AKO02, § 1.3]). *Let $\mathcal{C}$ be an additive category, let $A, B \in \mathrm{Ob}\left(\mathcal{C}\right)$ be two objects that admit decompositions into indecomposable objects, and let $\mathcal{I} \leq \mathcal{C}$ be an ideal. Let $A = \bigoplus_k A_k$ and $B = \bigoplus B_\ell$ be decompositions into indecomposables. For any morphism $f : A \to B$, with decomposition $f = \bigoplus_{k,\ell} f_{k\ell}$, we have*

$$f \in \mathcal{I}(A,B) \text{ if and only if } f_{k\ell} \in \mathcal{I}(A_k, B_\ell) \text{ for all } k, \ell. \tag{5.29}$$

Proof. Let $\mathrm{inc}_{A_k} : A_k \to A, \mathrm{inc}_{B_\ell} : B_\ell \to B, \mathrm{proj}_{A_k} : A \to A_k, \mathrm{proj}_{B_\ell} : B \to B_\ell$ be the inclusion and projection morphisms of the decompositions into indecomposables.

Suppose first that $f \in \mathcal{I}(A,B)$. As $\mathcal{I}$ is an ideal, we immediately find that $f_{k\ell} = \mathrm{proj}_{B_\ell} \circ f \circ \mathrm{inc}_{A_k} \in \mathcal{I}(A_k, B_\ell)$ for any k, ℓ.

Suppose that $f_{k\ell} \in \mathcal{I}(A_k, B_\ell)$ for all k, ℓ. $f = \sum_{k,\ell} \mathrm{inc}_{B_\ell} \circ f_{k\ell} \circ \mathrm{proj}_{A_k}$ is the sum of morphisms in the ideal composed with other morphisms, hence an element of the ideal. ■

Crucially, the indecomposable objects of the quotient category are fully determined by the indecomposable objects of the original category.

Proposition 5.2.6. *Let $\mathcal{C}$ be an additive category, and let $\mathcal{I} \leq \mathcal{C}$ be an ideal.*

1. *An object in $\mathcal{C}/\mathcal{I}$ is indecomposable if and only if it is the image of an indecomposable object in $\mathcal{C}$ under the canonical quotient functor $\mathrm{quot}_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}/\mathcal{I}$.*
2. *If $\mathcal{C}$ is semisimple and Schur, then $\mathrm{quot}_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ preserves simple objects (although non-zero simple objects could become null objects). As a consequence, an object in $\mathcal{C}/\mathcal{I}$ is simple if and only if it is the image of a simple object in $\mathcal{C}$ under $\mathrm{quot}_{\mathcal{I}}$.*
3. *If $\mathcal{C}$ is artinian abelian, an object in $\mathcal{C}/\mathcal{I}$ is indecomposable if and only if its endomorphism ring is local and all endomorphisms are either nilpotent or an isomorphism.*

Proof. For (1), suppose that $A \in \mathrm{Ob}(\mathcal{C})$ is indecomposable in $\mathcal{C}/\mathcal{I}$, but not in $\mathcal{C}$. This implies that we find indecomposable objects $X_1, \ldots, X_n \in \mathrm{Ob}(\mathcal{C})$ such that $A = X_1 \oplus \cdots \oplus X_n$ in $\mathcal{C}$. As $\mathrm{quot}_{\mathcal{I}}$ is additive, this then implies that $\mathrm{quot}_{\mathcal{I}}(A) = \mathrm{quot}_{\mathcal{I}}(X_1) \oplus \cdots \oplus \mathrm{quot}_{\mathcal{I}}(X_n)$. $\mathrm{quot}_{\mathcal{I}}(A)$ is indecomposable, which implies without loss of generality that $\mathrm{quot}_{\mathcal{I}}(X_2), \ldots, \mathrm{quot}_{\mathcal{I}}(X_n) = 0$, and thus $\mathrm{quot}_{\mathcal{I}}(A) = \mathrm{quot}_{\mathcal{I}}(X_1)$. This shows that it is the image of an indecomposable object under $\mathrm{quot}_{\mathcal{I}}$.

For the other direction and (3), it suffices to prove that $\mathrm{Hom}_{\mathcal{C}}(A, A)\,/\mathcal{I}(A, A)$ is local if $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is local (Proposition 5.1.12). This follows from Proposition 5.1.4 and Corollary 5.1.10.

For (2), let $A \in \mathrm{Ob}(\mathcal{C})$ be a simple object. For any morphism $i + \mathcal{I} : X \to A$ in $\mathcal{C}/\mathcal{I}$, the fact that $\mathcal{C}$ is semisimple and Schur implies that $i : X \to A$ is either zero or a split epimorphism (by composing with inc_B for any simple object B in a decomposition into simple objects of X). This then implies that $i + \mathcal{I}$ is also a split epimorphism. So, if $i + \mathcal{I} : X \to A$ is a monomorphism, then it is an isomorphism. We conclude that A is simple in $\mathcal{C}/\mathcal{I}$. The final part of the statement follows from (1). ■

Corollary 5.2.7. *Let $\mathcal{C}$ be a semisimple Schur category, and let $\mathcal{I} \leq \mathcal{C}$ be an ideal. $\mathcal{C}/\mathcal{I}$ is also a semisimple Schur category.*

Proof. Proposition 5.2.6 shows that any indecomposable object in $\mathcal{C}/\mathcal{I}$ is simple, and that any object $A \in \mathrm{Ob}\left(\mathcal{C}/\mathcal{I}\right)$ is simple if and only if it is the image of a simple object in $\mathcal{C}$. We conclude that $\mathcal{C}/\mathcal{I}$ is semisimple. Let $f + \mathcal{I} : A \to B$ be any morphism between simple objects in $\mathcal{C}/\mathcal{I}$. Without loss of generality, we can assume that A and B are simple in $\mathcal{C}$ too. This implies that $f : A \to B$ is either zero or an isomorphism, which implies that $f + \mathcal{I}$ is either zero or an isomorphism. ■

5.2.3 The radical of a category

The most important example (for us) of an ideal in a category enriched over some commutative ring, is a generalisation of the Jacobson radical 5.1.1.

Definition 5.2.8 (The Jacobson radical of a category, [AKO02, Définition 1.4.1]). Let R be a commutative ring, and let $\mathcal{C}$ be a ${}_R\mathbf{Mod}$-enriched category. For any $A, B \in \mathrm{Ob}(\mathcal{C})$, we define

$$\begin{aligned}
\mathrm{rad}(\mathcal{C})(A, B) :&= \{f : A \to B \mid (\forall g : B \to A)(\mathrm{id}_A - g \circ f \text{ is a split epimorphism})\} && (5.30)\\
&= \{f : A \to B \mid (\forall g : B \to A)(\mathrm{id}_A - g \circ f \text{ is an isomorphism})\} && (5.31)\\
&= \{f : A \to B \mid (\forall g : B \to A)(\mathrm{id}_B - f \circ g \text{ is a split monomorphism})\} && (5.32)\\
&= \{f : A \to B \mid (\forall g : B \to A)(\mathrm{id}_B - f \circ g \text{ is an isomorphism})\}. && (5.33)
\end{aligned}$$

The proof that this is is an ideal in the sense of Definition 5.2.1, and that the different definitions align, is exactly the same as the one given in Definition 5.1.1.

What the radical tells us about a category

A first interesting result is that the radical provides insight into a converse to Schur's lemma 2.4.9.

Proposition 5.2.9. *Let $\mathcal{C}$ be a Schur category, and let $A \in \mathrm{Ob}(\mathcal{C})$ be an artinian object such that $\mathrm{rad}(\mathcal{C})(B, A) = 0$ for every simple object $B \in \mathrm{Ob}(\mathcal{C})$. A is simple if and only if $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a division ring.*

Proof. Suppose that $\mathrm{Hom}_{\mathcal{C}}(B, A) \neq 0$ for some simple object $B \in \mathrm{Ob}(\mathcal{C})$. If $\mathrm{Hom}_{\mathcal{C}}(A, B) = 0$, then $\mathrm{rad}(\mathcal{C})(B, A) = \mathrm{Hom}_{\mathcal{C}}(B, A) \neq 0$. We conclude the result from Proposition 5.1.15. ■

In the context of division rings, we also have the following statement.

Lemma 5.2.10 ([AKO02, Lemme 1.4.9]). *Let $\mathcal{C}$ be a pre-additive category. If $A \in \mathrm{Ob}(\mathcal{C})$ is an object such that $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a division ring, then $\mathrm{rad}(\mathcal{C})(A, B) = \{f : A \to B \mid (\forall g : B \to A)(g \circ f = 0)\}$ for any other object $B \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Suppose that $f \in \mathrm{rad}(\mathcal{C})(A,B)$. We then know that for any morphism $g : B \to A$, $g \circ f \in \mathrm{rad}(\mathcal{C})(A,A)$. As $\mathrm{Hom}_{\mathcal{C}}(A,A)$ is a division ring, we know that $\mathrm{rad}(\mathcal{C})(A,A) = 0$, hence that $g \circ f = 0$. Proving the other inclusion is trivial. ∎

Finally, the radical tells us whether a category is semisimple.

Corollary 5.2.11. *Let $\mathcal{C}$ be an additive category.*

1. *If $\mathrm{rad}(\mathcal{C}) = 0$ and all endomorphism rings corresponding to indecomposable objects are local, then $\mathcal{C}$ is Schur.*
2. *If $\mathcal{C}$ is such that every object has a decomposition into indecomposable objects and such that all endomorphism rings corresponding to indecomposable objects are local, then $\mathcal{C}$ is Schur and semisimple if and only if $\mathrm{rad}(\mathcal{C}) = 0$.*

Proof. We will first prove (1). Let $A, B \in \mathrm{Ob}(\mathcal{C})$ be two indecomposable objects and let $f : A \to B$ be non-zero. We know that $\mathrm{rad}(\mathcal{C})(A,B) = 0$, which implies that there exists $g : B \to A$ such that $\mathrm{id}_A - g \circ f$ is not invertible. However, this implies that $\mathrm{id}_A - g \circ f$ is in the maximal ideal of the local ring $\mathrm{Hom}_{\mathcal{C}}(A,A)$, which is $\mathrm{rad}(\mathcal{C})(A,A) = 0$. We conclude that $g \circ f = \mathrm{id}_A$, and thus that f is a split monomorphism. Similarly we show that f is a split epimorphism, and we conclude that f is an isomorphism.

Suppose now that $\mathcal{C}$ is such that every object has a decomposition into indecomposable objects. We already know that $\mathcal{C}$ is Schur if $\mathrm{rad}(\mathcal{C}) = 0$, and Proposition 5.2.9 implies that all indecomposable objects in $\mathcal{C}$ are actually simple. We conclude that $\mathcal{C}$ is semisimple. Conversely, if $\mathcal{C}$ is Schur and semisimple, then Lemma 5.2.10 implies that $\mathrm{rad}(\mathcal{C})(A,B) = 0$ for any two indecomposable objects $A, B \in \mathrm{Ob}(\mathcal{C})$. We conclude that $\mathrm{rad}(\mathcal{C}) = 0$ through Proposition 5.2.5. ∎

Remark 5.2.12. Combining Proposition 5.2.9 and Corollary 5.2.11 gives Corollary 5.1.16.

Radical functors and the structure of quotients

The radical is sufficiently important that we introduce structure-preserving functors which respect it.

Definition 5.2.13 (Radical functors, [AKO02, Définition 1.4.6]). Let R be a commutative ring, and let $\mathcal{C}, \mathcal{D}$ be ${}_R\mathbf{Mod}$-enriched categories. An R-linear functor $F : \mathcal{C} \to \mathcal{D}$ is called *radical* if

$$F(\mathrm{rad}(\mathcal{C})) \subseteq \mathrm{rad}(\mathcal{D}), \text{ i.e. } F(\mathrm{rad}(\mathcal{C})(A,B)) \subseteq \mathrm{rad}(\mathcal{D})(F(A),F(B)) \text{ for all } A, B \in \mathrm{Ob}(\mathcal{C}). \quad (5.34)$$

Lemma 5.2.14 ([AKO02, Lemme 1.4.7]). *Let $\mathcal{C}, \mathcal{D}$ be pre-additive categories.*

1. *Full additive functors $F : \mathcal{C} \to \mathcal{D}$ are radical.*
2. *Additive functors $F : \mathcal{C} \to \mathcal{D}$ that reflect split epimorphisms are such that $F^{-1}(\mathrm{rad}(\mathcal{D})) \subseteq \mathrm{rad}(\mathcal{C})$.*
3. *Additive functors $F : \mathcal{C} \to \mathcal{D}$ that are full and reflect split epimorphisms are such that $F(\mathrm{rad}(\mathcal{C})) = \mathrm{rad}(\mathcal{D})|_{F(\mathrm{Ob}(\mathcal{C}))}$.*

Proof. Let $A, B \in \mathrm{Ob}(\mathcal{C})$. For the first statement, we want to show that $f \in \mathrm{rad}(\mathcal{C})(A,B)$ implies that $F(f) \in \mathrm{rad}(\mathcal{D})(F(A),F(B))$. Let $h \in \mathrm{Hom}_{\mathcal{D}}\big(F(B),F(A)\big)$, we have to show that $\mathrm{id}_{F(A)} - h \circ F(f)$ is a split epimorphism if $f \in \mathrm{rad}(C)(A,B)$. As F is full, we find $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ such that $F(g) = h$. It is then clear that $\mathrm{id}_{F(A)} - h \circ F(f) = F(\mathrm{id}_A - g \circ f)$ is a split epimorphism, as functors preserve split epimorphisms.

For the second statement, we want to show that if $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is such that $F(f) \in \mathrm{rad}(\mathcal{D})(F(A),F(B))$, then $f \in \mathrm{rad}(\mathcal{C})(A,B)$. For any $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$, we then have that $F(\mathrm{id}_A - g \circ f) = \mathrm{id}_{F(A)} - F(g) \circ F(f)$ is a split epimorphism, and thus that $\mathrm{id}_A - g \circ f$ is a split epimorphism. We conclude that $f \in \mathrm{rad}(\mathcal{C})(A,B)$.

The third statement follows from the first two statements. ∎

Lemma 5.2.15. *Let $\mathcal{C}$ be a pre-additive category. The canonical quotient functor* $\mathrm{quot}_{\mathrm{rad}(\mathcal{C})} : \mathcal{C} \to \mathcal{C}/\mathrm{rad}(\mathcal{C})$ *is full and reflects split epimorphisms and split monomorphisms, and thus in particular isomorphisms. As a consequence, we find* $\mathrm{rad}(\mathcal{C}/\mathrm{rad}(\mathcal{C})) = 0$.

Proof. Let $A, B \in \mathrm{Ob}\,(\mathcal{C})$, and suppose that $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is such that $\mathrm{quot}_{\mathrm{rad}(\mathcal{C})}(f)$ is a split epimorphism (resp. split monomorphism). This means that there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $f \circ g + \mathrm{rad}(\mathcal{C})(B, B) = \mathrm{id}_B + \mathrm{rad}(\mathcal{C})(B, B)$ (resp. $g \circ f + \mathrm{rad}(\mathcal{C})(A, A) = \mathrm{id}_A + \mathrm{rad}(\mathcal{C})(A, A)$), hence that there exists $k \in \mathrm{rad}(\mathcal{C})(B, B)$ (resp. $k \in \mathrm{rad}(\mathcal{C})(A, A)$) such that $f \circ g = \mathrm{id}_B - k$ (resp. $g \circ f = \mathrm{id}_A - k$). The definition of the radical implies that this is an isomorphism. We can thus find $h \in \mathrm{Hom}_{\mathcal{C}}(B, B)$ (resp. $h \in \mathrm{Hom}_{\mathcal{C}}(A, A)$) such that $f \circ (g \circ h) = \mathrm{id}_B$ (resp. $(h \circ g) \circ f = \mathrm{id}_A$), which shows that f is a split epimorphism (resp. split monomorphism).

The final statement follows from the above Lemma 5.2.14. ∎

Proposition 5.2.16 ([AKO02, Lemme 2.1.5]). *Let $\mathcal{C}$ be an additive category such that all endomorphism rings corresponding to indecomposable objects are local, and such that every object has a decomposition into indecomposable objects. For any ideal $\mathcal{I} \leq \mathcal{C}$, we have* $\mathrm{rad}(\mathcal{C}/\mathcal{I}) = \mathrm{quot}_{\mathcal{I}}(\mathrm{rad}(\mathcal{C})) = \mathrm{rad}(\mathcal{C}) + \mathcal{I}$.

Proof. We have a full linear functor $\mathcal{C}/\mathcal{I} \to \mathcal{C}/(\mathrm{rad}(\mathcal{C}) + \mathcal{I})$, which means that we can use Lemma 5.2.14. For any $f \in \mathrm{rad}(\mathcal{C}/\mathcal{I})$, we thus find that $f + \mathrm{rad}(\mathcal{C}) + \mathcal{I} \in \mathrm{rad}(\mathcal{C}/(\mathrm{rad}(\mathcal{C}) + \mathcal{I}))$. Now, Corollary 5.2.11, Proposition 5.2.6, and Proposition 5.1.4 show that $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ is a semisimple Schur category. Corollary 5.2.7 then shows that $\mathcal{C}/(\mathrm{rad}(\mathcal{C}) + \mathcal{I})$ is also a semisimple Schur category, and using Corollary 5.2.11 we find that $\mathrm{rad}(\mathcal{C}/(\mathrm{rad}(\mathcal{C}) + \mathcal{I})) = 0$. We conclude that $f + \mathrm{rad}(\mathcal{C}) + \mathcal{I} = \mathrm{rad}(\mathcal{C}) + \mathcal{I}$ for all $f \in \mathrm{rad}(\mathcal{C}/\mathcal{I})$, and thus that $\mathrm{rad}(\mathcal{C}/\mathcal{I}) \subseteq \mathrm{rad}(C) + \mathcal{I}$. The other inclusion is trivial. ∎

5.2.4 Semisimplification of abelian categories

Proposition 5.2.17. *Let $\mathcal{C}$ be an additive category such that all objects have a decomposition into indecomposable objects and such that all indecomposable objects have local endomorphism rings. $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ is a semisimple Schur category such that* $\mathrm{rad}(\mathcal{C}/\mathrm{rad}(\mathcal{C})) = 0$.

Proof. This follows from Lemma 5.2.15 and Corollary 5.2.11. ∎

Corollary 5.2.18. *Let $\mathcal{C}$ be an additive category such that all endomorphism rings corresponding to indecomposable objects are local, and such that every object has a decomposition into indecomposable objects (e.g. an abelian category in which every object has finite length). $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ is a semisimple Schur category. Furthermore, if $\mathcal{I} \leq \mathcal{C}$ is any ideal such that $\mathcal{C}/\mathcal{I}$ is semisimple and Schur, then* $\mathrm{rad}(\mathcal{C}) \subseteq \mathcal{I}$.

Proof. This follows from the above Proposition 5.2.17 Corollary 5.2.11, and Proposition 5.2.16. ∎

Remark 5.2.19. The above result shows that $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ is the semisimplification of $\mathcal{C}$ in the sense of Definition 5.0.2.

This naturally raises the question of what the structure of the quotient category $\mathcal{C}/\mathcal{I}$ is when $\mathrm{rad}(\mathcal{C}) \subseteq \mathcal{I}$.

Lemma 5.2.20. *Let $\mathcal{C}$ be an additive category such that all endomorphism rings corresponding to indecomposable objects are local, and such that every object has a decomposition into indecomposable objects (e.g. an abelian category in which every object has finite length). Let $\mathcal{I} \leq \mathcal{C}$ be any ideal, and let $A, B \in \mathrm{Ob}\,(\mathcal{C})$ be any two indecomposable objects. If $\mathcal{I}(A, B)$ is not contained in $\mathrm{rad}(\mathcal{C})(A, B)$, then $A \cong B$ and $\mathcal{I}(A, X) = \mathrm{Hom}_{\mathcal{C}}(A, X)\,, \mathcal{I}(X, A) = \mathrm{Hom}_{\mathcal{C}}(X, A)\,, \mathcal{I}(B, X) = \mathrm{Hom}_{\mathcal{C}}(B, X)\,, \mathcal{I}(X, B) = \mathrm{Hom}_{\mathcal{C}}(X, B)$ for all $X \in \mathrm{Ob}\,(\mathcal{C})$.*

This implies that any indecomposable object $A \in \mathrm{Ob}\,(\mathcal{C})$ satisfying this property (by which we mean that there exists an indecomposable object $B \in \mathrm{Ob}\,(\mathcal{C})$ such that $\mathcal{I}(A, B) \not\subseteq \mathrm{rad}(\mathcal{C})(A, B)$ or $\mathcal{I}(B, A) \not\subseteq \mathrm{rad}(\mathcal{C})(B, A)$) gets mapped to a null object under the canonical quotient functor $\mathrm{quot}_{\mathcal{I}}$.

Proof. Suppose that $f \in \mathcal{I}(A, B)$ but not in $\mathrm{rad}(\mathcal{C})(A, B)$. This implies that there exists $g, h : B \to A$ such that $\mathrm{id}_A - g \circ f$ and $\mathrm{id}_B - f \circ h$ are not invertible. However, as A and B are indecomposable we find that $\mathrm{Hom}_{\mathcal{C}}(A, A)$ and $\mathrm{Hom}_{\mathcal{C}}(B, B)$ are local rings, and thus that $g \circ f$ and $f \circ h$ are isomorphisms. We conclude that f is an isomorphism. This implies that $\mathrm{id}_A \in \mathcal{I}(A, A)$ and $\mathrm{id}_B \in \mathcal{I}(B, B)$ as $\mathrm{id}_A = f^{-1} \circ f$ and $\mathrm{id}_B = f \circ f^{-1}$. We conclude that any morphism $g : A \to X$ is contained in $\mathcal{I}(A, X)$ by composing with id_A, and similarly for the other equalities. ■

It is then clear that, when $\mathrm{rad}(\mathcal{C}) \subseteq \mathcal{I}$, $\mathcal{C}/\mathcal{I}$ is essentially $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ with some more simple objects set to zero. If $\mathcal{I}(A, B) \neq \mathrm{rad}(\mathcal{C})(A, B)$ for simple objects $A, B \in \mathrm{Ob}(\mathcal{C})$, then A and B are null objects. If $\mathcal{I}(A, B) = \mathrm{rad}(\mathcal{C})(A, B)$, then we find the same structure as in $\mathcal{C}/\mathrm{rad}(\mathcal{C})$.

When is the semisimplification simple

Above we provided a semisimplification procedure on $\mathbf{Struct} = \mathbf{FinAbCat}$, abelian categories in which all objects have finite length. However, we cannot assume at this point that this semisimplification procedure ends up in $\mathbf{FinAbCat}$.

More generally, there are two major pitfalls to be aware of when dealing with quotients of abelian categories. First, the quotient of an abelian category by an ideal is generally not abelian. Second, even when the quotient is abelian, the canonical quotient functor is almost never exact.

We will now prove that the semisimplification procedure does end up in a semisimple abelian category.

The proof of the following result was inspired by the author's earlier work in [Sle24, Proposition 7.3.1].

Proposition 5.2.21. *A semisimple Schur category $\mathcal{C}$ is abelian.*

Proof. Let $A, B \in \mathrm{Ob}(\mathcal{C})$ be objects with decompositions $A = A_1 \oplus \cdots \oplus A_n$ and $B = B_1 \oplus \cdots \oplus B_m$ into simple objects, and let $f : A \to B$ be a morphism. As $\mathcal{C}$ is Schur, we know that $f_{ij} := \mathrm{proj}_{B_j} \circ f \circ \mathrm{inc}_{A_i}$ is either zero or invertible for all i, j.

We will first prove that $\mathcal{C}$ is pre-abelian, i.e. that f admits a kernel and a cokernel.

Let X be a simple object that appears in the decomposition of A n_X and m_X times respectively (m_X could be zero), and let $\mathrm{inc}^i_X, \mathrm{proj}^j_X$ be the different inclusions of X into A and the different projections of B onto X respectively.

If $m_X = 0$, then we set $k^X := \mathrm{id}_{n_X X}$.

If $m_X \geq 1$, then we define an $m_X \times n_X$-matrix over the division ring $R = \mathrm{Hom}_{\mathcal{C}}(X, X)$

$$f^{(X)} := \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n_X 1} \\ f_{12} & f_{22} & \cdots & f_{n_X 2} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1 m_X} & f_{2 m_X} & \cdots & f_{n_X m_X} \end{bmatrix} \text{ where } f_{ij} := \mathrm{proj}^j_X \circ f \circ \mathrm{inc}^i_X\,. \tag{5.35}$$

This is a morphism in the abelian category of finitely generated R-modules, ${}_R\mathbf{FinMod}$, and we thus know that it has a kernel $\ker(f^{(X)})$. Because R is a division ring, we know that every R-module is free. In particular, this implies that $\ker(f^{(X)}) : R^{a_X} \to R^{n_X}$ for some $a_X \geq 0$. This then induces a morphism $k^X : a_X X \to n_X X$ as $k^X := \sum_{i=1}^{a_X} \sum_{j=1}^{n_X} \mathrm{inc}_j \circ \ker(f^{(X)})_{ij} \circ \mathrm{proj}_i$. We know that, as a monomorphism in a semisimple abelian category, $\ker(f^{(X)})$ is split (Proposition 2.4.7). This implies that k^X is a split monomorphism.

We then define $k := \sum_X \mathrm{inc}_{n_X X} \circ k^X \circ \mathrm{proj}_{a_X X}$, which is once again a split monomorphism because it decomposes into split monomorphisms.

We have

$$
\begin{aligned}
f \circ k &= \sum_X \sum_{i,j,k} \mathrm{inc}_X^j \circ f_{ij} \circ k_{ki}^X \circ \mathrm{proj}_X^k \\
&= \sum_X \mathrm{inc}_{n_X X} \circ f^{(X)} \cdot \ker(f^{(X)}) \circ \mathrm{proj}_{a_X X}, \\
&= 0
\end{aligned}
\tag{5.36}
$$

where we have used the fact that the contributions from f are zero when X is not a direct summand in both the decompositions of A and B (as $\mathcal{C}$ is Schur).

Let g be any other morphism such that $f \circ g = 0$. Suppose that $g : C \to A$ with a decomposition $C = C_1 \oplus \cdots \oplus C_r$. We know that any direct summand in the decomposition of C that does not appear in the decomposition of A will result in a zero contribution to g. So, assume that X is a direct summand in the decompositions of C and A. As $\ker(f^{(X)})$ is the kernel for $f^{(X)}$, we know that there is a uniquely induced matrix $\ell^{(X)}$ such that $\ker(f^{(X)}) \cdot \ell^{(X)} = g^{(X)}$ (if X does not appear in the decomposition of B then this is trivial with $\ker(f^{(X)})$ the identity matrix). These matrices induce a morphism ℓ such that $k \circ \ell = g$ (which can be checked as in the above), necessarily unique as k is a monomorphism.

We will now prove that all monomorphisms are split monomorphisms, and similarly we would find that all epimorphisms are split epimorphisms. Proposition 2.2.5 then implies that monomorphisms are kernels and that epimorphisms are cokernels, from which we can conclude that $\mathcal{C}$ is abelian.

Suppose that $f : A \to B$ is a monomorphism. If there is a simple object X in the decomposition of A that does not appear in the decomposition of B (i.e. such that $m_X = 0$), then f cannot be a monomorphism as $f \circ \mathrm{inc}_X = 0$.

Furthermore, as f is a monomorphism, the matrices $f^{(X)}$ are monomorphisms too. As monomorphisms in a semisimple abelian category, these are then split monomorphisms. We conclude that f is a split monomorphism as before. ■

Corollary 5.2.22. *Let $\mathcal{C}$ be an additive category such that all objects have a decomposition into indecomposable objects and such that all indecomposable objects have local endomorphism rings. $\mathcal{C}/\mathrm{rad}(\mathcal{C})$ is a semisimple abelian category. Moreover, any ideal $\mathcal{I} \leq \mathcal{C}$ such that $\mathcal{C}/\mathcal{I}$ is semisimple and abelian must contain* $\mathrm{rad}(\mathcal{C})$.

Proof. This follows from Proposition 5.2.17 and Proposition 5.2.21. ■

5.3 Ideals in tensor categories

5.3.1 Tensor ideals

Above, we discussed ideals in ${}_R\mathbf{Mod}$-enriched categories, which made sense as these are algebroids. We can do something similar for monoidal categories, as the monoidal product equips morphisms with some notion of multiplication[3]. This leads to the following notion of ideals in monoidal categories.

Definition 5.3.1 (Tensor ideals in monoidal categories, [EO21a, § 2.1]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category that is pre-additive, and suppose that the monoidal product is bilinear on morphisms (Proposition 4.1.1 and Proposition 4.1.2 then show that the category is enriched over the commutative ring $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$). A *tensor ideal* $\mathcal{I}$ in $\mathcal{C}$ consists of an ideal $\mathcal{I} \leq \mathcal{C}$ in the pre-additive sense (these will automatically be ideals for the ring $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$), such that for all $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathcal{I}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(C, D)$

$$
f \otimes g \in \mathcal{I}(A \otimes C, B \otimes D) \text{ and } g \otimes f \in \mathcal{I}(C \otimes A, D \otimes B). \tag{5.37}
$$

If only the first of those two properties holds, then $\mathcal{I}$ is called a *right tensor ideal*, and if only the second of those two properties holds, then $\mathcal{I}$ is called a *left tensor ideal.*

[3] One could think of an additive monoidal category with bilinear monoidal product as being like rings on two levels: the composition gives endomorphism sets a ring structure, while biproducts and the monoidal product endow objects with something resembling a ring structure.

Equipped with such a tensor ideal $\mathcal{I}$, the quotient category $\mathcal{C}/\mathcal{I}$ defined in Definition 5.2.3 is again a pre-additive monoidal[4] category (the monoidal product is inherited from the monoidal product on $\mathcal{C}$). In this case, the canonical quotient functor $\mathrm{quot}_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ is monoidal by setting $\zeta := \mathrm{id}_{\mathbb{1}} + \mathcal{I}(\mathbb{1}, \mathbb{1})$ and $\varepsilon_{(A,B)} := \mathrm{id}_{A \otimes B} + \mathcal{I}(A \otimes B, A \otimes B)$. This functor allows us to transfer a lot of knowledge and data from $\mathcal{C}$ to $\mathcal{C}/\mathcal{I}$ (duals, braidings, ...).

Remark 5.3.2. These ideals are instances of Definition 5.0.1, with $\mathbf{Struct}$ consisting of pre-additive monoidal categories such that the monoidal product is bilinear on morphisms, $\mathcal{M} = \mathbf{Ab}$, and $\mathrm{BinOp} = \{\circ, \otimes\}$.

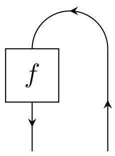
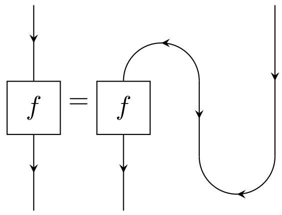
Remark 5.3.3 *([AKO02, Définition 6.1.1]).* An equivalent definition of tensor ideals is that a pre-additive ideal $\mathcal{I} \leq \mathcal{C}$ is a tensor ideal if, for every object $A \in \mathrm{Ob}(\mathcal{C})$, the maps $\mathrm{id}_A \otimes -$ and $- \otimes \mathrm{id}_A$ send morphisms in $\mathcal{I}$ to morphisms in $\mathcal{I}$.

5.3.2 Constructing tensor ideals

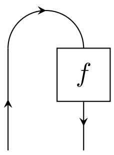
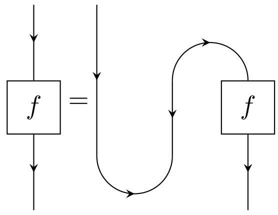
We will now work towards finding the largest proper right or left tensor ideal in pre-additive left or right rigid monoidal categories such that the monoidal product is bilinear on morphisms, as discussed in [AKO02].

Proposition 5.3.4 ([AKO02, Lemme 6.1.5]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and pre-additive category such that the monoidal product is bilinear on morphisms. For two right (resp. left) tensor ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{C}$, we have $\mathcal{I} \subseteq \mathcal{J}$ (i.e. $\mathcal{I}(A,B) \subseteq \mathcal{J}(A,B)$ for all $A, B \in \mathrm{Ob}(\mathcal{C})$) if and only if $\mathcal{I}(\mathbb{1}, A) \subseteq \mathcal{J}(\mathbb{1}, A)$ for all $A \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are right tensor ideals, suppose that the second property holds, and suppose that we have some $f \in \mathcal{I}(A,B)$ with $A, B \in \mathrm{Ob}(\mathcal{C})$. Then, because $\mathcal{I}$ is a right tensor ideal


$\in \mathcal{I}(\mathbb{1}, B \otimes A^*) \subseteq \mathcal{J}(\mathbb{1}, B \otimes A^*)$, and thus

$\in \mathcal{J}(A,B)$. (5.38)

Similarly, for left tensor ideals in right rigid categories


$\in \mathcal{I}(\mathbb{1}, {}^*A \otimes B) \subseteq \mathcal{J}(\mathbb{1}, {}^*A \otimes B)$, and thus

$\in \mathcal{J}(A,B)$. (5.39)

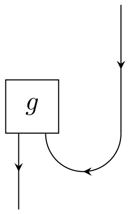
■

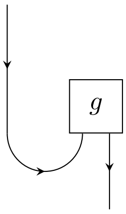
The preceding result suggests a method for constructing a right tensor ideal and a left tensor ideal from a family of subgroups $\mathcal{I}(\mathbb{1}, A) \leq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$.

Construction 1. Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and pre-additive category such that the monoidal product is bilinear on morphisms. If we have a family of subgroups $\mathcal{I}(A) \leq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ for all $A \in \mathrm{Ob}(\mathcal{C})$,

[4]If we use left or right tensor ideals, then we will not end up with a monoidal category, but with a left or right module category over the original monoidal category.

which are closed under left composition in the sense that $g \in \mathcal{I}(A)$ implies $k \circ g \in \mathcal{I}(B)$ for all $k \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then we can construct an ideal $\mathcal{I}_{\otimes}^{\mathrm{left}}$ (resp. $\mathcal{I}_{\otimes}^{\mathrm{right}}$), where for $A, B \in \mathrm{Ob}\,(\mathcal{C})$

$\mathcal{I}_{\otimes}^{\mathrm{left}}(A, B)$ contains  for $g \in \mathcal{I}(B \otimes A^*)$, and (5.40)

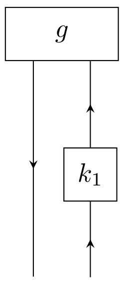
$\mathcal{I}_{\otimes}^{\mathrm{right}}(A, B)$ contains  for $g \in \mathcal{I}({}^*A \otimes B)$. (5.41)

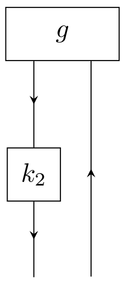
Note that $\mathcal{I}_{\otimes}^{\mathrm{left}}(\mathbb{1}, A) = \mathcal{I}(A)$ (resp. $\mathcal{I}_{\otimes}^{\mathrm{right}}(\mathbb{1}, A) = \mathcal{I}(A)$).

Proposition 5.3.5 ([AKO02, Lemme 6.1.6]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and pre-additive category such that the monoidal product is bilinear on morphisms. If we have a class of subgroups $\mathcal{I}$ given by $\mathcal{I}(A) \leq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ for all $A \in \mathrm{Ob}\,(\mathcal{C})$, and if $\mathcal{I}$ is closed under left composition in the sense that $g \in \mathcal{I}(A)$ implies $k \circ g \in \mathcal{I}(B)$ for all $k \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then $\mathcal{I}_{\otimes}^{\mathrm{left}}$ (resp. $\mathcal{I}_{\otimes}^{\mathrm{right}}$), defined in Construction 1 above, is a right (resp. left) tensor ideal in $\mathcal{C}$.*

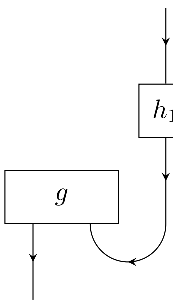
Proof. We will only prove the statement for left rigid categories.

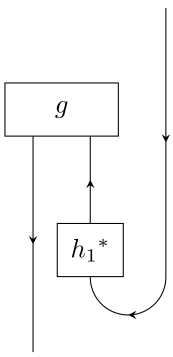
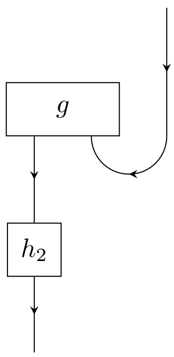
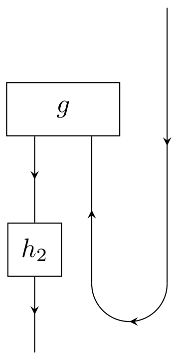
We will first prove that $\mathcal{I}_{\otimes}^{\mathrm{left}}$ is an ideal in the sense of Definition 5.2.1, i.e. a pre-additive ideal. For this, note that for all $g \in \mathcal{I}(B \otimes A^*)$, $k_1 \in \mathrm{Hom}_{\mathcal{C}}(A^*, C^*)$, $k_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$



$\in \mathcal{I}(B \otimes C^*)$ and  $\in \mathcal{I}(C \otimes A^*)$. (5.42)

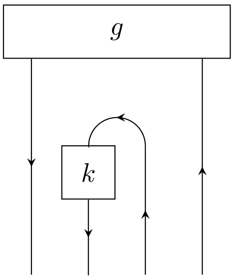
This implies that for arbitrary $h_1 \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ and $h_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$



$=$  $\in \mathcal{I}_{\otimes}^{\mathrm{left}}(C, B)$ and  $=$  $\in \mathcal{I}_{\otimes}^{\mathrm{left}}(A, C)$.

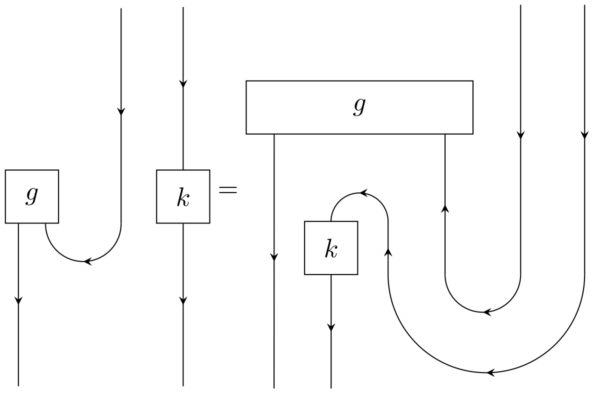
(5.43)

We now need to prove that $\mathcal{I}_{\otimes}^{\mathrm{left}}$ is a right ideal in the sense of Definition 5.3.1, i.e. that is it is closed under monoidal products. For this, note that for all $g \in \mathcal{I}(B \otimes A^*), k \in \mathrm{Hom}_{\mathcal{C}}(C, D)$



$$\in \mathcal{I}(B \otimes D \otimes D^* \otimes B^*), \tag{5.44}$$

and thus



$$\in \mathcal{I}_{\otimes}^{\text{left}}(A \otimes C, B \otimes D). \tag{5.45}$$

■

Remark 5.3.6. Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and pre-additive category such that the monoidal product is bilinear on morphisms, and let $\mathcal{I} \leq \mathcal{C}$ be an arbitrary pre-additive ideal in the sense of Definition 5.2.1. Construction 1 defines an ideal $\mathcal{I}_{\otimes}^{\text{left}}$ (resp. $\mathcal{I}_{\otimes}^{\text{right}}$) by applying the construction to the groups $\mathcal{I}(\mathbb{1}, A)$.

By Proposition 5.3.4 and Proposition 5.3.5, $\mathcal{I}$ coincides with $\mathcal{I}_{\otimes}^{\text{left}}$ (resp. $\mathcal{I}_{\otimes}^{\text{right}}$) when $\mathcal{I}$ is already a left (resp. right) tensor ideal. Even stronger, for two pre-additive ideals $\mathcal{I}, \mathcal{J} \leq \mathcal{C}$, an inclusion $\mathcal{I}(\mathbb{1}, A) \subseteq \mathcal{J}(\mathbb{1}, A)$ (e.g. when $\mathcal{I} \subseteq \mathcal{J}$) suffices to show that $\mathcal{I}_{\otimes}^{\text{left}} \subseteq \mathcal{J}_{\otimes}^{\text{left}}$ (resp. $\mathcal{I}_{\otimes}^{\text{right}} \subseteq \mathcal{J}_{\otimes}^{\text{right}}$).

5.3.3 Maximal tensor ideals

We now aim to show that tensor categories are "local" in the sense that they resemble local rings. To this end, we seek an appropriate notion of a radical in tensor categories. A natural candidate in our setting arises by applying Construction 1 to the pre-additive Jacobson radical $\mathrm{rad}(\mathcal{C})$.

In the next proposition, which is a slight generalisation of [AKO02, Proposition 7.1.4], we demonstrate that this notion of a radical generalises (1) and (2) in Proposition 5.1.2.

Proposition 5.3.7 ([AKO02, Proposition 7.1.4]). *Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and pre-additive category such that the monoidal product is bilinear on morphisms, and such that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field (equivalently, due to Propositon 4.1.1, a division ring). The right (resp. left) tensor ideal $\mathrm{rad}(\mathcal{C})_{\otimes}^{\text{left}}$ (resp. $\mathrm{rad}(\mathcal{C})_{\otimes}^{\text{right}}$) is the maximal proper right (resp. left) tensor ideal of $\mathcal{C}$.*

Suppose that, in addition, $\mathcal{C}$ is additive, such that all objects have a decomposition into indecomposable objects, and such that all endomorphism rings corresponding to indecomposable objects are local. If $\mathcal{I} \leq \mathcal{C}$ is a right (resp. left) tensor ideal such that $\mathcal{C}/\mathcal{I}$ is semisimple, Schur, and non-zero, then $\mathcal{I} = \mathrm{rad}(\mathcal{C})_{\otimes}^{\text{left}}$ (resp. $\mathcal{I} = \mathrm{rad}(\mathcal{C})_{\otimes}^{\text{right}}$).

Proof. We will only prove the statement for right tensor ideals; the proof for left tensor ideals is analogous.

Proposition 5.3.4 and Remark 5.3.6 show that we only have to prove that for any left tensor ideal $\mathcal{I}$, we have $\mathcal{I}(\mathbb{1}, A) \subseteq \mathrm{rad}(\mathcal{C})(\mathbb{1}, A)$ for all $A \in \mathrm{Ob}(\mathcal{C})$. Applying Lemma 5.2.10, we find that $f \in \mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(\mathbb{1}, A) = \mathrm{rad}(\mathcal{C})(\mathbb{1}, A)$ if and only if $g \circ f = 0$ for all $g : A \to \mathbb{1}$. For $f \in \mathcal{I}(\mathbb{1}, A)$, suppose that there exists $g : A \to \mathbb{1}$ such that $g \circ f \neq 0$. Because $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field, we may assume that $g \circ f = \mathrm{id}_{\mathbb{1}}$. This implies that $\mathrm{id}_{\mathbb{1}} \in \mathcal{I}(\mathbb{1}, \mathbb{1})$, and thus that $\mathcal{I}(\mathbb{1}, A) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$. Proposition 5.3.4 now implies that $\mathcal{I} = \mathrm{Hom}(\mathcal{C})$.

Suppose now that $\mathcal{C}$ is additive, such that all objects have a decomposition into indecomposable objects, and such that all endomorphism rings corresponding to indecomposable objects are local. If $\mathcal{I}$ is such that $\mathcal{C}/\mathcal{I}$ is semisimple and Schur, then $\mathrm{rad}(\mathcal{C}) \subseteq \mathcal{I}$ through Corollary 5.2.18. Remark 5.3.6 then implies that $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes} \subseteq \mathcal{I}$. As $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ is maximal, and $\mathcal{C}/\mathcal{I}$ is non-zero (hence $\mathcal{I} \neq \mathrm{Hom}(\mathcal{C})$), this implies that $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes} = \mathcal{I}$. ■

Remark 5.3.8. We did not prove that $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ (resp. $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes}$) is semisimple and Schur when $\mathcal{C}$ is well-behaved with regard to indecomposable objects. Rather, we showed that if there exists a right (resp. left) tensor ideal $\mathcal{I} \subsetneq \mathcal{C}$ such that the quotient $\mathcal{C}/\mathcal{I}$ is semisimple (such an ideal need not exist a priori), then $\mathcal{I}$ must be equal to $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ (resp. $\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes}$). In Corollary 5.6.1, we will show that these quotients are indeed semisimple and Schur when the category is rigid.

The above generalisation of (1) and (2) in Proposition 5.1.2 raises a natural question: do the maximal tensor ideals also satisfy the property of local rings that the maximal ideals consist of the non-invertible morphisms? The answer is, roughly, yes. More precisely, in Proposition 5.5.10, we will show that this holds for indecomposable objects that are not "of dimension zero".

5.3.4 Negligible morphisms

We now know that tensor categories admit maximal tensor ideals, but we do not yet have a concrete description of their structure. In Section 5.5, we will show that these ideals roughly consist of the non-invertible morphisms between indecomposable objects. However, in the case of pivotal categories, we will show in this section that these ideals admit particularly nice explicit descriptions in terms of the left and right traces introduced in Section 3.5.

Definition 5.3.9 (Tensor ideals of negligible morphisms, [EO21a, Definition 2.1]). Let $\mathcal{C}$ be a pre-additive pivotal category (with pivotal structure α) such that the monoidal product is bilinear on morphisms. We construct ideals

$$\mathcal{N}^{\mathrm{left}}(A, B) := \{f : A \to B \mid (\forall g : B \to A)(\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ g \circ f) = 0)\}, \tag{5.46}$$

$$\mathcal{N}^{\mathrm{right}}(A, B) := \{f : A \to B \mid (\forall g : B \to A)(\mathrm{tr}^{\mathrm{right}}(g \circ f \circ \alpha_A^{-1}) = 0)\}, \tag{5.47}$$

called the *ideal of left negligible morphisms* and the *ideal of right negligible morphisms* respectively.

Proposition 3.5.4 shows that these can equivalently be defined as

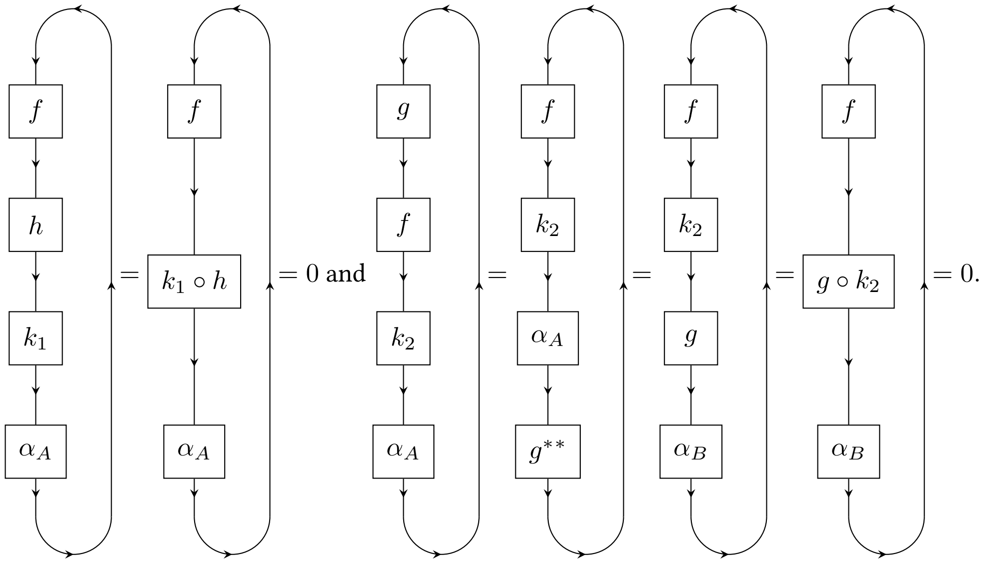
$$\mathcal{N}^{\mathrm{left}}(A, B) := \{f : A \to B \mid (\forall g : B \to A)(\mathrm{tr}^{\mathrm{left}}(\alpha_B \circ f \circ g) = 0)\}, \tag{5.48}$$

$$\mathcal{N}^{\mathrm{right}}(A, B) := \{f : A \to B \mid (\forall g : B \to A)(\mathrm{tr}^{\mathrm{right}}(f \circ g \circ \alpha_B^{-1}) = 0)\}. \tag{5.49}$$

Proposition 5.3.10 ([AKO02, Lemme 7.1.1] and [EO21a, Lemma 2.3]). *Let $\mathcal{C}$ be a pre-additive pivotal category such that the monoidal product is bilinear on morphisms. The ideals of left and right negligible morphisms, defined in Definition 5.3.9, are right and left tensor ideals respectively.*

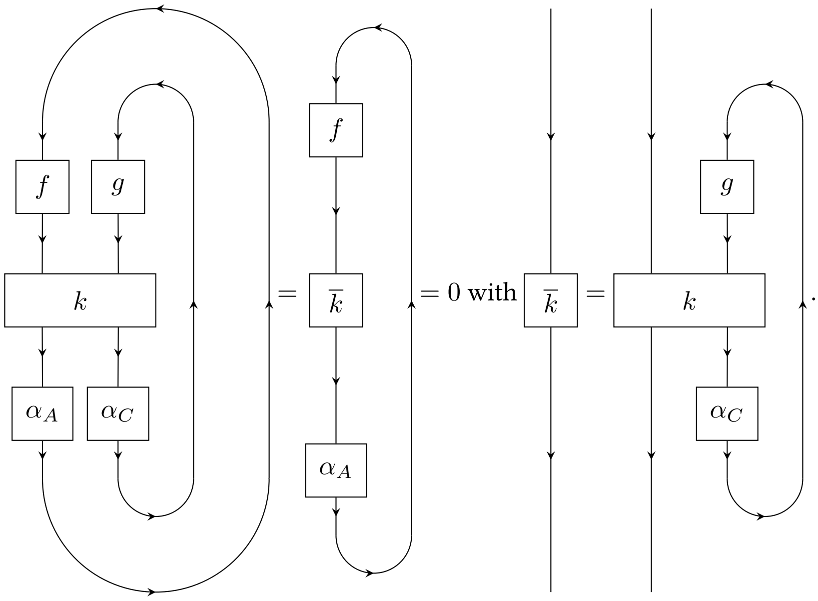
Proof. Through the additivity of the trace (which follows from the fact the composition and monoidal product are bilinear on morphisms), we see that $\mathcal{N}^{\mathrm{left}}(A, B)$ is a subgroup of $(\mathrm{Hom}_{\mathcal{C}}(A, B), +)$

To see that $\mathcal{N}^{\text{left}}$ is a pre-additive ideal in the sense of Definition 5.2.1, let $A, B, C, D \in \text{Ob}(\mathcal{C})$, let $g \in \text{Hom}_{\mathcal{C}}(A, B)$, $h \in \text{Hom}_{\mathcal{C}}(C, D)$, and let $f \in \mathcal{N}^{\text{left}}(B, C)$. We find that $h \circ f \in \mathcal{N}^{\text{left}}(B, D)$ and $f \circ g \in \mathcal{N}^{\text{left}}(A, C)$, as for any $k_1 : D \to A$ and $k_2 : C \to A$



(5.50)

Finally, we show that $\mathcal{N}^{\text{left}}$ is a right tensor ideal in the sense of Definition 5.3.1. Let $A, B, C, D \in \text{Ob}(\mathcal{C})$, let $g \in \text{Hom}_{\mathcal{C}}(C, D)$, and let $f \in \mathcal{N}^{\text{left}}(A, B)$. For any $k : B \otimes D \to A \otimes C$, we find



(5.51)

■

Remark 5.3.11. If the category $\mathcal{C}$ in Definition 5.3.9 is spherical, then $\mathcal{N}^{\text{left}} = \mathcal{N}^{\text{right}}$, and these are then tensor ideals. More generally, this holds when $\mathcal{C}$ is such that for all $f : A \to A$ in $\mathcal{C}$, we have $\text{tr}^{\text{left}}(\alpha_A \circ f) = 0$ if and only if $\text{tr}^{\text{right}}(f \circ \alpha_A^{-1}) = 0$.

We will now prove that these ideals are the maximal tensor ideals.

Proposition 5.3.12 ([AKO02, Proposition 7.1.4]). *Let $\mathcal{C}$ be a pre-additive pivotal category such that the monoidal product is bilinear on morphisms, and such that $\text{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field.*

We have

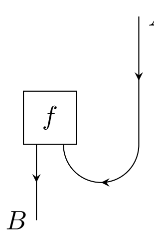
$$\mathcal{N}^{\text{left}} = \text{rad}(\mathcal{C})^{\text{left}}_{\otimes} \ (\textit{resp. } \mathcal{N}^{\text{right}} = \text{rad}(\mathcal{C})^{\text{right}}_{\otimes}). \tag{5.52}$$

As a consequence, the right (resp. left) tensor ideal of left (resp. right) negligible morphisms $\mathcal{N}^{\text{left}}$ (resp. $\mathcal{N}^{\text{right}}$) is the maximal proper right (resp. left) tensor ideal of $\mathcal{C}$.

Proof. We will only prove the statements for right tensor ideals. The proof for left tensor ideals is identical when considering the second equivalent definition of right negligible morphisms (5.49).

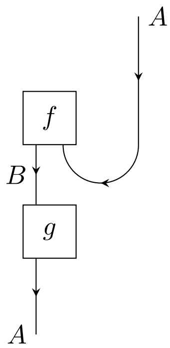
The result now follows from Example (34), Lemma 5.2.10, Proposition 5.3.4, and Proposition 5.3.5. The second statement follows from Proposition 5.3.7. ■

Remark 5.3.13. The use of Proposition 5.3.4 obscures the equality of the tensor ideals $\mathcal{N}^{\text{left}}$ and $\mathrm{rad}(\mathcal{C})^{\text{left}}_{\otimes}$, it is not immediately evident why one should expect these to be equal in general (although the above proof is of course valid). However, working with all of the morphisms of $\mathrm{rad}(\mathcal{C})^{\text{left}}_{\otimes}$ explicitly, we see why this is the case. Morphisms of $\mathrm{rad}(\mathcal{C})^{\text{left}}_{\otimes}(A, B)$ are of the form



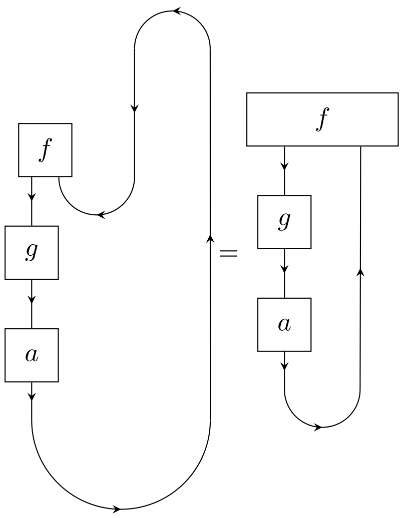
for $f : \mathbb{1} \to B \otimes A^* \in \mathrm{rad}(\mathcal{C})(\mathbb{1}, B \otimes A^*)$, (5.53)

which means that for any $g : B \to A$, we have



$\in \mathrm{rad}(\mathcal{C})^{\text{left}}_{\otimes}(A, A)$. (5.54)

Taking the trace after composing with some $a : A \to A^{**}$ results in



$\in \mathrm{rad}(\mathcal{C})(\mathbb{1}, \mathbb{1}) = 0$ because $\mathrm{rad}(\mathcal{C})(\mathbb{1}, \mathbb{1})$ is a proper ideal in a field.

(5.55)

Remark 5.3.14. As a corollary of the above Proposition 5.3.12, we find that the tensor ideals of negligible morphisms do not depend on the choice of pivotal structure (as tensor ideals are defined using only the pre-additive monoidal structure).

5.4 Properties of traces with regard to to an abelian structure

We now aim to work towards an intrinsic description of the morphisms lying in the maximal tensor ideals. To achieve this for the tensor ideals of negligible morphisms, we need a deeper understanding of how monoidal traces interact with the abelian structure.

By Proposition 5.2.5, it suffices to study the negligible morphisms between indecomposable objects. Furthermore, Proposition 5.1.9 tells us that endomorphisms of indecomposable objects are either isomorphisms or nilpotent. Consequently, it is essential to understand the traces of nilpotent morphisms.

5.4.1 Additivity of the trace on short exact sequences

Before we can discuss the trace of a nilpotent morphism, we have to understand the behaviour of monoidal traces on short exact sequences.

Lemma 5.4.1 ([Bra14, Lemma 3.1.20 (2)]). *Let $\mathcal{C}$ be an abelian monoidal category such that the monoidal product is biexact, and suppose that we have two short exact sequences*

$$\begin{array}{c} 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \\ \\ 0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0 \end{array} . \tag{5.56}$$

We have a short exact sequence

$$0 \longrightarrow \operatorname{Im}(\overline{f}) \xrightarrow{\operatorname{im}(\overline{f})} B \otimes B' \xrightarrow{g \otimes g'} C \otimes C' \longrightarrow 0 \ , \tag{5.57}$$

where $\overline{f} := (f \otimes \mathrm{id}_{B'}) \circ \mathrm{proj}_{A \otimes B'} + (\mathrm{id}_B \otimes f') \circ \mathrm{proj}_{B \otimes A'}$ ($\operatorname{Im}(\overline{f})$ is hence a quotient of $(A \otimes B') \oplus (B \otimes A')$).

Proof. We know that f, f' are monomorphisms, and that g, g' are epimorphisms. As the monoidal product is biexact, we then find that $f \otimes \mathrm{id}_X, \mathrm{id}_X \otimes f'$ are monomorphisms, and that $g \otimes \mathrm{id}_X, \mathrm{id}_X \otimes g'$ are epimorphisms for any $X \in \mathrm{Ob}(\mathcal{C})$. $\operatorname{im}(\overline{f})$ is trivially a monomorphism too, and $g \otimes g' = (g \otimes \mathrm{id}_{C'}) \circ (\mathrm{id}_B \otimes g')$ is an epimorphism.

The only thing left for us to prove is that $g \otimes g'$ is a cokernel of $\operatorname{im}(\overline{f})$, or equivalently $\overline{f}$.

Clearly, $(g \otimes g') \circ \overline{f} = 0$. Suppose now that $h : B \otimes B' \to X$ is such that $h \circ \overline{f} = 0$. Composing from the right with $\mathrm{inc}_{A \otimes B'}$ and $\mathrm{inc}_{B \otimes A'}$, we find $h \circ (f \otimes \mathrm{id}_{B'}) = 0$ and $h \circ (\mathrm{id}_B \otimes f') = 0$. As $g \otimes \mathrm{id}_{B'} = \operatorname{coker}(f \otimes \mathrm{id}_{B'})$, we find a unique $\overline{h}$ such that $\overline{h} \circ (g \otimes \mathrm{id}_{B'}) = h$.

It is then easy to show that $\overline{h} \circ (\mathrm{id}_B \otimes f') = 0$: $g \otimes \mathrm{id}_{A'}$ is an epimorphism, and we thus find $\overline{h} \circ (\mathrm{id}_B \otimes f') = 0$ if and only if $\overline{h} \circ (g \otimes f') = 0$, but $\overline{h} \circ (g \otimes \mathrm{id}_{B'}) \circ (\mathrm{id}_B \otimes f') = h \circ (\mathrm{id}_B \otimes f') = 0$.

As before, we can then find a unique $\underline{h}$ such that $\underline{h} \circ (\mathrm{id}_B \otimes g') = \overline{h}$. We finally obtain $h = \overline{h} \circ (g \otimes \mathrm{id}_{B'}) = \underline{h} \circ (g \otimes g')$. As $g \otimes g'$ is an epimorphism, $\underline{h}$ is unique with this property. ■

The proof of the following theorem was inspired by the proof of [GKP11, Lemma 2.5.1] (which was in turn inspired by the very short proof of [Del07, Lemme 3.5]), but this is an element-free (and more general) version.

Theorem 5.4.2 ([Del07, Lemme 3.5]). *Let $\mathcal{C}$ be an abelian left rigid monoidal category such that the monoidal product is bilinear and biexact (which is true if the category is also right rigid through Proposition 4.3.3) on morphisms. Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A^{**} & \xrightarrow{f^{**}} & B^{**} & \xrightarrow{g^{**}} & C^{**} & \longrightarrow & 0 \end{array} \tag{5.58}$$

be a morphism of short exact sequences in $\mathcal{C}$ (note that the second sequence is exact due to Corollary 3.4.9). We have

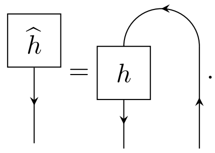
$$\mathrm{tr}^{\mathrm{left}}(b) = \mathrm{tr}^{\mathrm{left}}(a) + \mathrm{tr}^{\mathrm{left}}(c). \tag{5.59}$$

Proof. As the left dualisation functor $-^* : \mathcal{C}^{\mathrm{dual}} \to \mathcal{C}$ is left exact (it is actually exact, Proposition 4.3.1), we obtain a left exact sequence

$$0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \ . \tag{5.60}$$

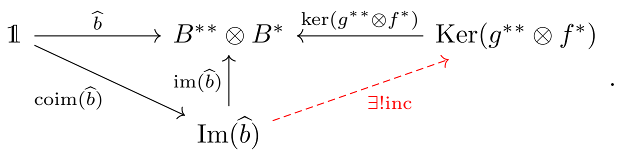
This means that g^* is a monomorphism, and that g^* is a kernel of f^*.

For $h : X \to X$ in $\mathcal{C}$, define $\widehat{h} : \mathbb{1} \to X \otimes X^*$ as



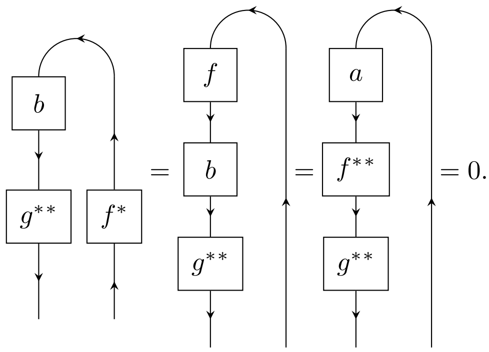
(5.61)

We claim that there is a unique morphism $\mathrm{inc} : \mathrm{Im}(\widehat{b}) \to \mathrm{Ker}(g^{**} \otimes f^*)$ such that the following diagram commutes



(5.62)

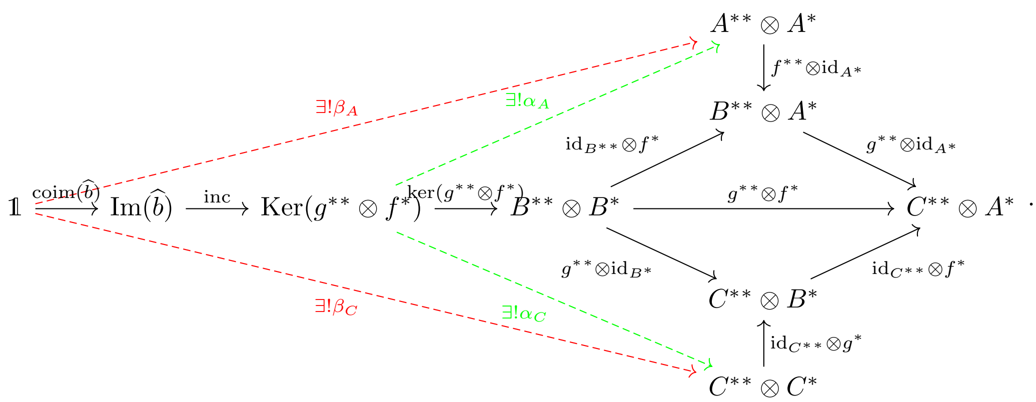
For this, we use Lemma 3.5.3 to obtain



(5.63)

As $\widehat{b} = \mathrm{im}(\widehat{b}) \circ \mathrm{coim}(\widehat{b})$, and $\mathrm{coim}(\widehat{b})$ is an epimorphism, we then find that $(g^{**} \otimes f^*) \circ \mathrm{im}(\widehat{b}) = 0$. This shows that inc does indeed exist, and is unique.

In the following diagram, α_A, β_A (resp. α_C, β_C) are induced by $\ker(\mathrm{id}_{C^{**}} \otimes f^*) = \mathrm{id}_{C^{**}} \otimes g^*$ (resp. $\ker(g^{**} \otimes \mathrm{id}_{A^*}) = f^{**} \otimes \mathrm{id}_{A^*}$). These dashed morphisms make the squares involving only one dashed morphism commute. As $f^{**} \otimes \mathrm{id}_{A^*}$ and $\mathrm{id}_{C^{**}} \otimes g^*$ are monomorphisms, the triangles involving two dashed morphisms commute too. We conclude that the following diagram is commutative



(5.64)

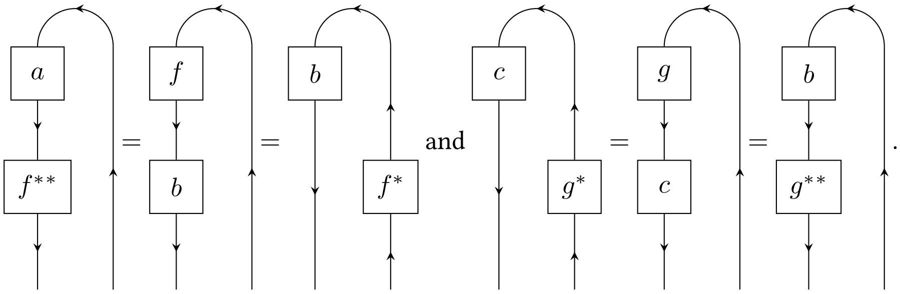
Lemma 5.4.1 shows that $\mathrm{Ker}(g^{**} \otimes f^*)$ is a quotient of $(\mathrm{Ker}(g^{**}) \otimes B^*) \oplus (B^{**} \otimes \mathrm{Ker}(f^*)) = (A^{**} \otimes B^*) \oplus (B^{**} \otimes C^*)$, and that $\ker(g^{**} \otimes f^*) = \mathrm{im}(\zeta)$ with $\zeta = (f^{**} \otimes \mathrm{id}_{B^*}) \circ \mathrm{proj}_{A^{**} \otimes B^*} + (\mathrm{id}_{B^{**}} \otimes g^*) \circ \mathrm{proj}_{B^{**} \otimes C^*}$. We then find

$$(f^{**} \otimes \mathrm{id}_{A^*}) \circ \alpha_A = (\mathrm{id}_{B^{**}} \otimes f^*) \circ \mathrm{im}(\zeta) \text{ and } (\mathrm{id}_{C^{**}} \otimes g^*) \circ \alpha_C = (g^{**} \otimes \mathrm{id}_{B^*}) \circ \mathrm{im}(\zeta), \tag{5.65}$$

which implies that

$$\alpha_A \circ \mathrm{coim}(\zeta) = (\mathrm{id}_{A^{**}} \otimes f^*) \circ \mathrm{proj}_{A^{**} \otimes B^*} \text{ and } \alpha_C \circ \mathrm{coim}(\zeta) = (g^{**} \otimes \mathrm{id}_{C^*}) \circ \mathrm{proj}_{B^{**} \otimes C^*}. \tag{5.66}$$

Note that $\beta_A = \widehat{a}$ and $\beta_C = \widehat{c}$ through Lemma 3.5.3 and (5.62)



(5.67)

We then obtain

$$\begin{aligned} \mathrm{tr}^{\mathrm{left}}(b) &= \mathrm{ev}_{B^*} \circ \widehat{b} \\ &= \mathrm{ev}_{B^*} \circ \ker(g^{**} \otimes f^*) \circ \mathrm{inc} \circ \mathrm{coim}(\widehat{b}). \\ &= \mathrm{ev}_{B^*} \circ \mathrm{im}(\zeta) \circ \mathrm{inc} \circ \mathrm{coim}(\widehat{b}) \end{aligned} \tag{5.68}$$

We have

$$\begin{aligned} \mathrm{ev}_{B^*} \circ \mathrm{im}(\zeta) \circ \mathrm{coim}(\zeta) &= \mathrm{ev}_{B^*} \circ ((f^{**} \otimes \mathrm{id}_{B^*}) \circ \mathrm{proj}_{A^{**} \otimes B^*} + (\mathrm{id}_{B^{**}} \otimes g^*) \circ \mathrm{proj}_{B^{**} \otimes C^*}) \\ &= (\mathrm{ev}_{A^*} \circ (\mathrm{id}_{A^{**}} \otimes f^*) \circ \mathrm{proj}_{A^{**} \otimes B^*} + \mathrm{ev}_{C^*} \circ (g^{**} \otimes \mathrm{id}_{C^*}) \circ \mathrm{proj}_{B^{**} \otimes C^*}) \\ &= (\mathrm{ev}_{A^*} \circ \alpha_A \circ \mathrm{coim}(\zeta) + \mathrm{ev}_{C^*} \circ \alpha_C \circ \mathrm{coim}(\zeta)) \\ &= (\mathrm{ev}_{A^*} \circ \alpha_A + \mathrm{ev}_{C^*} \circ \alpha_C) \circ \mathrm{coim}(\zeta) \end{aligned}, \tag{5.69}$$

where we used Lemma 3.5.3 and (5.66). We can thus conclude that $\mathrm{ev}_{B^*} \circ \mathrm{im}(\zeta) = \mathrm{ev}_{A^*} \circ \alpha_A + \mathrm{ev}_{C^*} \circ \alpha_C$. Plugging this into (5.68), and using the commutativity of (5.64), gives

$$\begin{aligned} \mathrm{tr}^{\mathrm{left}}(b) &= (\mathrm{ev}_{A^*} \circ \alpha_A + \mathrm{ev}_{C^*} \circ \alpha_C) \circ \mathrm{inc} \circ \mathrm{coim}(\widehat{b}) \\ &= \mathrm{ev}_{A^*} \circ \beta_A + \mathrm{ev}_{C^*} \circ \beta_C \\ &= \mathrm{ev}_{A^*} \circ \widehat{a} + \mathrm{ev}_{C^*} \circ \widehat{c} \\ &= \mathrm{tr}^{\mathrm{left}}(a) + \mathrm{tr}^{\mathrm{left}}(c) \end{aligned}. \tag{5.70}$$

■

5.4.2 Nilpotent morphisms have trace zero

Using the above result on the additivity of traces over short exact sequences, we can prove that the trace of a nilpotent endomorphism is zero. We will not prove this in full generality at this point (a complete treatment is deferred to Proposition 5.5.8), but we will follow the standard approach found in the literature.

Corollary 5.4.3 ([Del07, Corollaire 3.6]). *Let $\mathcal{C}$ be an abelian pivotal category (with pivotal structure α) such that the monoidal product is bilinear on morphisms (and thus biexact through Proposition 4.3.3), let $A \in \mathrm{Ob}(\mathcal{C})$ be an object, and let $f : A \to A$ be an endomorphism.*

If f is nilpotent, i.e. if there exists k such that $f^k = 0$, then $\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ f) = 0$.

Proof. As $\operatorname{coker}(f^n) \circ f^{n+1} = 0$, we find $\operatorname{coker}(f^n) \circ \operatorname{im}(f^{n+1}) = 0$ and $\operatorname{coker}(f^n) \circ f \circ \operatorname{im}(f^n) = 0$, which implies that there are uniquely induced morphisms $i_n : \operatorname{Im}(f^{n+1}) \to \operatorname{Im}(f^n)$ and $f_n : \operatorname{Im}(f^n) \to \operatorname{Im}(f^n)$ such that $\operatorname{im}(f^n) \circ i_n = \operatorname{im}(f^{n+1})$ and $\operatorname{im}(f^n) \circ f_n = f \circ \operatorname{im}(f^n)$. Similarly, $\operatorname{coker}(f^{n+1}) \circ f^{n+1} = 0$ implies that $\operatorname{coker}(f^{n+1}) \circ f \circ \operatorname{im}(f^n) = 0$, which implies that there is a uniquely induced morphism $\alpha_n : \operatorname{Im}(f^n) \to \operatorname{Im}(f^{n+1})$ such that $\operatorname{im}(f^{n+1}) \circ \alpha_n = f \circ \operatorname{im}(f^n)$. For these morphisms, we have

$$\begin{aligned} \operatorname{im}(f^n) \circ f_n \circ i_n &= f \circ \operatorname{im}(f^n) \circ i_n \\ &= f \circ \operatorname{im}(f^{n+1}) \\ &= \operatorname{im}(f^{n+1}) \circ f_{n+1} \\ &= \operatorname{im}(f^n) \circ i_n \circ f_{n+1} \end{aligned} , \tag{5.71}$$

$$\operatorname{im}(f^n) \circ i_n \circ \alpha_n = \operatorname{im}(f^{n+1}) \circ \alpha_n = f \circ \operatorname{im}(f^n) = \operatorname{im}(f^n) \circ f_n, \tag{5.72}$$

$$\begin{aligned} \operatorname{im}(f^{n+1}) \circ \alpha_n \circ i_n &= f \circ \operatorname{im}(f^n) \circ i_n \\ &= f \circ \operatorname{im}(f^{n+1}) \\ &= \operatorname{im}(f^{n+1}) \circ f_{n+1} \end{aligned} . \tag{5.73}$$

We thus conclude that the following diagram commutes

$$\begin{tikzcd} \cdots \arrow[r, "i_4"] & \operatorname{Im}(f^4) \arrow[r, "i_3"] \arrow[d, "f_4"] \arrow[dl, "\alpha_4"'] & \operatorname{Im}(f^3) \arrow[r, "i_2"] \arrow[d, "f_3"] \arrow[dl, "\alpha_3"'] & \operatorname{Im}(f^2) \arrow[r, "i_1"] \arrow[d, "f_2"] \arrow[dl, "\alpha_2"'] & \operatorname{Im}(f) \arrow[r, "i_0"] \arrow[d, "f_1"] \arrow[dl, "\alpha_1"'] & A \arrow[d, "f"] \arrow[dl, "\alpha_0"'] \\ \cdots \arrow[r, "i_4"] & \operatorname{Im}(f^4) \arrow[r, "i_3"] & \operatorname{Im}(f^3) \arrow[r, "i_2"] & \operatorname{Im}(f^2) \arrow[r, "i_1"] & \operatorname{Im}(f) \arrow[r, "i_0"] & A \end{tikzcd} . \tag{5.74}$$

For each n, this commutative diagram induces morphisms of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Im}(f^{n+1}) \arrow[r, "i_n"] \arrow[d, "f_{n+1}"] & \operatorname{Im}(f^n) \arrow[r, "\operatorname{coker}(i_n)"] \arrow[d, "f_n"] & \operatorname{Coker}(i_n) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\ 0 \arrow[r] & \operatorname{Im}(f^{n+1}) \arrow[r, "i_n"] \arrow[d, "\alpha_{\operatorname{Im}(f^{n+1})}"] & \operatorname{Im}(f^n) \arrow[r, "\operatorname{coker}(i_n)"] \arrow[d, "\alpha_{\operatorname{Im}(f^n)}"] & \operatorname{Coker}(i_n) \arrow[r] \arrow[d, "\alpha_{\operatorname{Coker}(i_n)}"] & 0 \\ 0 \arrow[r] & \operatorname{Im}(f^{n+1})^{**} \arrow[r, "i_n^{**}"] & \operatorname{Im}(f^n)^{**} \arrow[r, "\operatorname{coker}(i_n)^{**}"] & \operatorname{Coker}(i_n)^{**} \arrow[r] & 0 \end{tikzcd} , \tag{5.75}$$

where the top right uniquely induced morphism is zero as $\operatorname{coker}(i_n) \circ f_n = \operatorname{coker}(i_n) \circ i_n \circ \alpha_n = 0$.

Theorem 5.4.2 now implies that $\operatorname{tr}^{\text{left}}(\alpha_{\operatorname{Im}(f^n)} \circ f_n) = \operatorname{tr}^{\text{left}}(\alpha_{\operatorname{Im}(f^{n+1})} \circ f_{n+1})$, from which we conclude

$$\operatorname{tr}^{\text{left}}(\alpha_A \circ f) = \operatorname{tr}^{\text{left}}(\alpha_{\operatorname{Im}(f^k)} \circ f_k) = \operatorname{tr}^{\text{left}}(0) = 0. \tag{5.76}$$

■

5.5 Classification of the morphisms in the maximal tensor ideals

In this section, we provide intrinsic descriptions of the morphisms in the maximal tensor ideals. We begin with the setting of pivotal categories commonly found in the literature, and then move on to the more general setting of possibly non-pivotal tensor categories.

Remark 5.5.1. In the literature (see, for example, [EO21a]), these results are typically stated for symmetric categories. However, Theorem 3.6.11 and Corollary 3.6.12 show that symmetric categories are pivotal and spherical. This implies that the results for symmetric categories follow as special cases of our more general discussion for pivotal categories.

5.5.1 Classification of negligible morphisms

The statements (and their proof) in this subsection are based on the statement and proof of [EO21a, Lemma 2.2].

Lemma 5.5.2. *Let $\mathcal{C}$ be a Karoubian category, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be objects such that A (resp. B) is indecomposable. If a morphism $f : A \to B$ is not an isomorphism, then for any $C \in \mathrm{Ob}(\mathcal{C})$ and any $g : C \to A$ (resp. $g : B \to C$), $f \circ g$ (resp. $g \circ f$) is not an isomorphism either.*

Proof. If $f \circ g$ (resp. $g \circ f$) is an isomorphism, then f is a split epimorphism from an indecomposable object (resp. a split monomorphism into an indecomposable object).

Proposition 2.2.5 now shows that A is decomposable, which is a contradiction. ∎

Proposition 5.5.3 ([EO21a, Lemma 2.2]). *Let $\mathcal{C}$ be an abelian pivotal category such that the monoidal product is bilinear on morphisms, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be indecomposable objects.*

1. *We have*

$$\{f : A \to B \mid f \text{ not invertible}\} \subseteq \mathcal{N}^{\mathrm{left}}(A, B) \subseteq \frac{\{f : A \to B \mid \dim^{\mathrm{left}}(A) = 0 \text{ or } f \text{ not invertible}\}}{\{f : A \to B \mid \dim^{\mathrm{left}}(B) = 0 \text{ or } f \text{ not invertible}\}}, \tag{5.77}$$

$$\{f : A \to B \mid f \text{ not invertible}\} \subseteq \mathcal{N}^{\mathrm{right}}(A, B) \subseteq \frac{\{f : A \to B \mid \dim^{\mathrm{right}}(A) = 0 \text{ or } f \text{ not invertible}\}}{\{f : A \to B \mid \dim^{\mathrm{right}}(B) = 0 \text{ or } f \text{ not invertible}\}}. \tag{5.78}$$

2. *If $\mathcal{C}$ is, in addition, $\mathbf{Vect}_{\mathbb{K}}$-enriched, with $\mathbb{K}$ some algebraically closed field, then we obtain*

$$\mathcal{N}^{\mathrm{left}}(A, B) = \frac{\{f : A \to B \mid \dim^{\mathrm{left}}(A) = 0 \text{ or } f \text{ is not invertible}\}}{\{f : A \to B \mid \dim^{\mathrm{left}}(B) = 0 \text{ or } f \text{ is not invertible}\}}, \tag{5.79}$$

$$\mathcal{N}^{\mathrm{right}}(A, B) = \frac{\{f : A \to B \mid \dim^{\mathrm{right}}(A) = 0 \text{ or } f \text{ not invertible}\}}{\{f : A \to B \mid \dim^{\mathrm{right}}(B) = 0 \text{ or } f \text{ not invertible}\}}. \tag{5.80}$$

Proof. We will only prove the statements for left negligible morphisms, the proof for right negligible morphisms is analogous.

For the first inclusion, we use Lemma 5.5.2 to find that $g \circ f$ is not an isomorphism if f is not an isomorphism, hence that $g \circ f$ is nilpotent through Proposition 5.1.9, which implies that $\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ g \circ f) = 0$ through Corollary 5.4.3.

For the second inclusion, suppose that $f \in \mathcal{N}^{\mathrm{left}}(A, B)$ is an isomorphism. This then implies that $\dim^{\mathrm{left}}(A) = \mathrm{tr}^{\mathrm{left}}(\alpha_A \circ f^{-1} \circ f) = 0$ and $\dim^{\mathrm{left}}(B) = \mathrm{tr}^{\mathrm{left}}(\alpha_B \circ f \circ f^{-1}) = 0$ (through the equivalent definitions for left negligible morphisms).

Suppose now that $\mathcal{C}$ is $\mathbf{Vect}_{\mathbb{K}}$-enriched, with $\mathbb{K}$ some algebraically closed field. For any $f : A \to B$ and $g : B \to A$, Corollary 5.1.11 states that $g \circ f = \lambda_g \mathrm{id}_A + h$ for some $\lambda_g \in \mathbb{K}$ and some nilpotent morphism $h : A \to A$. As a consequence, we find that $\mathrm{tr}^{\mathrm{left}}(\alpha_A \circ g \circ f) = \lambda_g \dim^{\mathrm{left}}(A) = 0$ (resp. $\mathrm{tr}^{\mathrm{left}}(\alpha_B \circ f \circ g) = \lambda \dim^{\mathrm{left}}(B) = 0$) if and only if either $\dim^{\mathrm{left}}(A) = 0$ (resp. $\dim^{\mathrm{left}}(B) = 0$), or if $\lambda_g = 0$. If f is an isomorphism, then $\lambda_{f^{-1}} = 1 \neq 0$, which implies that $f \in \mathcal{N}^{\mathrm{left}}(A, B)$ if and only if $\dim^{\mathrm{left}}(A) = 0$. ∎

Corollary 5.5.4. *Let $\mathcal{C}$ be an abelian pivotal category such that the monoidal product is bilinear on morphisms, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be objects of finite length with (unique due to the Krull-Schmidt theorem 2.4.4) decompositions into indecomposable objects $A = \bigoplus_k A_k$ and $B = \bigoplus_\ell B_\ell$. For morphisms $f : A \to B$, this implies that we have decompositions $f = \bigoplus_{k,\ell} f_{k\ell}$ with $f_{k\ell} = \mathrm{proj}_{B_\ell} \circ f \circ \mathrm{inc}_{A_k} : A_k \to B_\ell$.*

1. *We have*

$$\mathcal{N}^{\text{left}}(A,B) \subseteq \frac{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{left}}(A_k) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{left}}(B_\ell) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}, \tag{5.81}$$

$$\mathcal{N}^{\text{right}}(A,B) \subseteq \frac{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{right}}(A_k) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{right}}(B_\ell) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}. \tag{5.82}$$

2. *If $\mathcal{C}$ is, in addition, $\mathbf{Vect}_{\mathbb{K}}$-enriched, with $\mathbb{K}$ some algebraically closed field, then the above inclusions become equalities*

$$\mathcal{N}^{\text{left}}(A,B) = \frac{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{left}}(A_k) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{left}}(B_\ell) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}, \tag{5.83}$$

$$\mathcal{N}^{\text{right}}(A,B) = \frac{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{right}}(A_k) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}{\{f : A \to B \mid (\forall k,\ell)(\dim^{\text{right}}(B_\ell) = 0 \textit{ or } f_{k\ell} \textit{ not invertible})\}}. \tag{5.84}$$

Proof. This follows from Proposition 5.2.5 and Proposition 5.5.3. ■

5.5.2 Classification of radical morphisms in non-pivotal categories

Inspired by the above, we want to give a similar classification of morphisms in $\operatorname{rad}(\mathcal{C})^{\text{left}}_{\otimes}$ and $\operatorname{rad}(\mathcal{C})^{\text{right}}_{\otimes}$, which we call *left* or *right radical morphisms*. Ideally, one that works when we are not working over algebraically closed fields too.

First, we want to get rid of the pivotal structure, and work directly with morphisms $A \to A^{**}$. Note that, in rigid categories, $-^{**}$ is an equivalence, which implies that A^{**} is indecomposable if A is. This already equips us with some of the important tools used in the above (e.g. Lemma 5.5.2).

Another important tool we wish to use is Proposition 5.1.9. To apply this result in our context, we must first make sense of nilpotent morphisms to double duals.

Definition 5.5.5 (Nilpotent morphisms to double duals). Let $\mathcal{C}$ be a left (resp. right) rigid monoidal and pre-additive category, and let $A \in \operatorname{Ob}(\mathcal{C})$ be an object. For any morphism $f : A \to A^{**}$ (resp. any morphism $f : A \to {}^{**}A$), and any n, we define $f^{(0)} := \operatorname{id}_A$ and $f^{(n)} := f^{((n-1)**)} \circ f^{((n-2)**)} \circ \cdots \circ f^{**} \circ f : A \to A^{(n**)}$ (resp. ${}^{(0)}f := \operatorname{id}_A$ and ${}^{(n)}f := {}^{((n-1)**)}f \circ {}^{((n-2)**)}f \circ \cdots \circ {}^{**}f \circ f : A \to {}^{(n**)}A$). f is called *nilpotent* if there exists k such that $f^{(k)} = 0$ (resp. ${}^{(k)}f = 0$).

Remark 5.5.6. This is just the notion of nilpotence introduced in Definition 5.1.5 applied to $F = -^{**}, {}^{**}-$.

Proposition 5.5.7. *Let $\mathcal{C}$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms, let $A \in \operatorname{Ob}(\mathcal{C})$ be an indecomposable object of finite length, and let $f : A \to A^{**}$ be a morphism. Then either f is an isomorphism, or f is nilpotent.*

Proof. This follows by setting $F = -^{**}$ in Proposition 5.1.8. ■

Even though we are no longer considering negligible morphisms, the trace will still play a role in our classification of radical morphisms. We therefore seek a slight generalisation of Corollary 5.4.3.

Proposition 5.5.8. *Let $\mathcal{C}$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms, let $A \in \operatorname{Ob}(\mathcal{C})$ be an object, and let $f : A \to A^{**}$ be a morphism. If f is nilpotent, in the sense that there exists some k such that $f^{(k)} = 0$, then $\operatorname{tr}^{\text{left}}(f) = 0$.*

Proof. As in the proof of Corollary 5.4.3, we have a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{{}^{**}i_4} & \mathrm{Im}({}^{4**}f^{(4)}) & \xrightarrow{{}^{**}i_3} & \mathrm{Im}({}^{3**}f^{(3)}) & \xrightarrow{{}^{**}i_2} & \mathrm{Im}({}^{2**}f^{(2)}) & \xrightarrow{{}^{**}i_1} & \mathrm{Im}({}^{**}f) & \xrightarrow{{}^{**}i_0} & A \\
& \swarrow^{\alpha_4} & \downarrow f_4 & \swarrow^{\alpha_3} & \downarrow f_3 & \swarrow^{\alpha_2} & \downarrow f_2 & \swarrow^{\alpha_1} & \downarrow f_1 & \swarrow^{\alpha_0} & \downarrow f \\
\cdots & \xrightarrow{i_4} & \mathrm{Im}({}^{3**}f^{(4)}) & \xrightarrow{i_3} & \mathrm{Im}({}^{2**}f^{(3)}) & \xrightarrow{i_2} & \mathrm{Im}({}^{**}f^{(2)}) & \xrightarrow{i_1} & \mathrm{Im}(f) & \xrightarrow{i_0} & A^{**}
\end{array} . \tag{5.85}$$

1. We have $\mathrm{coker}({}^{((n-1)**)}f^{(n)}) \circ {}^{(n**)}f^{(n+1)} = 0$, which implies that also $\mathrm{coker}({}^{((n-1)**)}f^{(n)}) \circ \mathrm{im}({}^{(n**)}f^{(n+1)}) = 0$. We thus find a uniquely induced $i_n : \mathrm{Im}({}^{(n**)}f^{(n+1)}) \to \mathrm{Im}({}^{((n-1)**)}f^{(n)})$ such that
$$\mathrm{im}({}^{((n-1)**)}f^{(n)}) \circ i_n = \mathrm{im}({}^{(n**)}f^{(n+1)}). \tag{5.86}$$

2. Similarly, we have $\mathrm{coker}({}^{(n**)}f^{(n+1)}) \circ {}^{(n**)}f^{(n+1)} = 0$, which implies that $\mathrm{coker}({}^{(n**)}f^{(n+1)}) \circ f \circ \mathrm{im}({}^{(n**)}f^{(n)}) = 0$. This induces a unique morphism $\alpha_n : \mathrm{Im}({}^{(n**)}f^{(n)}) \to \mathrm{Im}({}^{(n**)}f^{(n+1)})$ such that
$$\mathrm{im}({}^{(n**)}f^{(n+1)}) \circ \alpha_n = f \circ \mathrm{im}({}^{(n**)}f^{(n)}). \tag{5.87}$$

3. Finally, we have $\mathrm{coker}({}^{((n-1)**)}f^{(n)}) \circ {}^{(n**)}f^{(n+1)} = 0$, which implies that $\mathrm{coker}({}^{((n-1)**)}f^{(n)}) \circ f \circ \mathrm{im}({}^{(n**)}f^{(n)}) = 0$. This induces a unique morphism $f_n : \mathrm{Im}({}^{(n**)}f^{(n)}) \to \mathrm{Im}({}^{((n-1)**)}f^{(n)})$ such that
$$\mathrm{im}({}^{((n-1)**)}f^{(n)}) \circ f_n = f \circ \mathrm{im}({}^{(n**)}f^{(n)}). \tag{5.88}$$

The diagram (5.85) commutes as

1.
$$\begin{aligned}
\mathrm{im}({}^{((n-1)**)}f^{(n)}) \circ f_n \circ {}^{**}i_n &= f \circ \mathrm{im}({}^{(n**)}f^{(n)}) \circ {}^{**}i_n \\
&= f \circ \mathrm{im}({}^{((n+1)**)}f^{(n+1)}) \\
&= \mathrm{im}({}^{(n**)}f^{(n+1)}) \circ f_{n+1} \\
&= \mathrm{im}({}^{(n-1)**}f^{(n)}) \circ i_n \circ f_{n+1}
\end{aligned} , \tag{5.89}$$
which implies that $f_n \circ {}^{**}i_n = i_n \circ f_{n+1}$,

2.
$$\begin{aligned}
\mathrm{im}({}^{(n-1)**}f^{(n)}) \circ i_n \circ \alpha_n &= \mathrm{im}({}^{(n**)}f^{(n+1)}) \circ \alpha_n \\
&= f \circ \mathrm{im}({}^{(n**)}f^{(n)}) \\
&= \mathrm{im}({}^{((n-1)**)}f^{(n)}) \circ f_n
\end{aligned} , \tag{5.90}$$
which implies that $i_n \circ \alpha_n = f_n$,

3.
$$\begin{aligned}
\mathrm{im}({}^{(n**)}f^{(n+1)}) \circ \alpha_n \circ {}^{**}i_n &= f \circ \mathrm{im}({}^{n**}f^{(n)}) \circ {}^{**}i_n \\
&= f \circ \mathrm{im}({}^{((n+1)**)}f^{(n+1)}), \\
&= \mathrm{im}({}^{(n**)}f^{(n+1)}) \circ f_{n+1}
\end{aligned} \tag{5.91}$$
which implies that $\alpha_n \circ {}^{**}i_n = f_{n+1}$.

For every n, this then induces a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}({}^{((n+1)**)}f^{(n+1)}) & \xrightarrow{{}^{**}i_n} & \mathrm{Im}({}^{(n**)}f^{(n)}) & \xrightarrow{\mathrm{coker}({}^{**}i_n)} & \mathrm{Coker}({}^{**}i_n) & \longrightarrow & 0 \\
& & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow \exists! & & \\
0 & \longrightarrow & \mathrm{Im}({}^{((n**)}f^{(n+1)}) & \xrightarrow{i_n} & \mathrm{Im}({}^{((n-1)**)}f^{(n)}) & \xrightarrow{\mathrm{coker}(i_n)} & \mathrm{Coker}(i_n) & \longrightarrow & 0
\end{array} , \tag{5.92}$$

where the uniquely induced morphism $\mathrm{Coker}({}^{**}i_n) \to \mathrm{Coker}(i_n)$ is zero as $\mathrm{coker}(i_n) \circ f_n = \mathrm{coker}(i_n) \circ i_n \circ \alpha_n = 0$.

Theorem 5.4.2 then implies that $\mathrm{tr}^{\mathrm{left}}(f_n) = \mathrm{tr}^{\mathrm{left}}(f_{n+1})$ for all n, and thus that

$$\mathrm{tr}^{\mathrm{left}}(f) = \mathrm{tr}^{\mathrm{left}}(f_k) = 0. \tag{5.93}$$

■

Remark 5.5.9. Proposition 5.5.8 implies Corollary 5.4.3 by setting $\overline{f} := \alpha_A \circ f$ there (as then $f^{(n)} = \left(\alpha_A^{((n-1)**)}\right)^n \circ f^n = 0$ if and only if $f^n = 0$).

We are now ready to prove a classification of left and right radical morphisms, which is very similar to Proposition 5.5.3.

Proposition 5.5.10. *Let $\mathcal{C}$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be indecomposable objects.*

$$\mathrm{rad}(\mathcal{C})_\otimes^{\mathrm{left}}(A,B) = \left\{ f : A \to B \,\middle|\, f \text{ not invertible or } (\forall a : A \to A^{**}) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{left}}(a) \text{ invertible}\right) \right\} \tag{5.94}$$

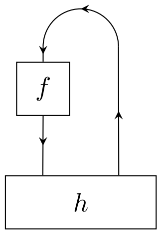
$$= \left\{ f : A \to B \,\middle|\, f \text{ not invertible or } (\forall b : B \to B^{**}) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{left}}(b) \text{ invertible}\right) \right\}, \tag{5.95}$$

$$\mathrm{rad}(\mathcal{C})_\otimes^{\mathrm{right}}(A,B) = \left\{ f : A \to B \,\middle|\, f \text{ not invertible or } (\forall a : A \to {}^{**}A) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{right}}(a) \text{ invertible}\right) \right\} \tag{5.96}$$

$$= \left\{ f : A \to B \,\middle|\, f \text{ not invertible or } (\forall b : B \to {}^{**}B) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{right}}(b) \text{ invertible}\right) \right\}. \tag{5.97}$$

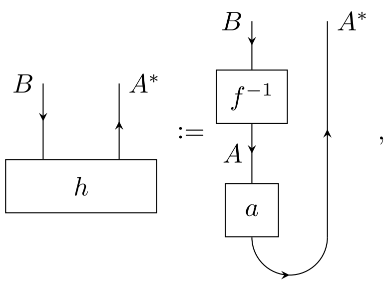
Proof. We will only prove the first equality, the proofs for the other equalities are similar.

Let us first prove that $\mathrm{rad}(\mathcal{C})_\otimes^{\mathrm{left}}(A,B)$ is included in the set on the right. Suppose that $f \in \mathrm{rad}(\mathcal{C})_\otimes^{\mathrm{left}}(A,B)$, and that f is invertible. As $\widehat{f} = (f \otimes \mathrm{id}_{A^*}) \circ \mathrm{coev}_A \in \mathrm{rad}(\mathcal{C})_\otimes^{\mathrm{left}}(\mathbb{1}, B \otimes A^*) = \mathrm{rad}(\mathcal{C})(\mathbb{1}, B \otimes A^*)$, we find that

$\mathrm{id}_{\mathbb{1}} -$ 

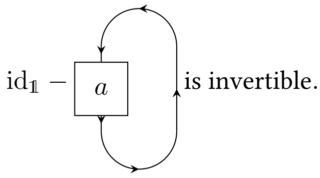
is invertible for any $h : B \otimes A^* \to \mathbb{1}$. (5.98)

Setting, for arbitrary $a : A \to A^{**}$



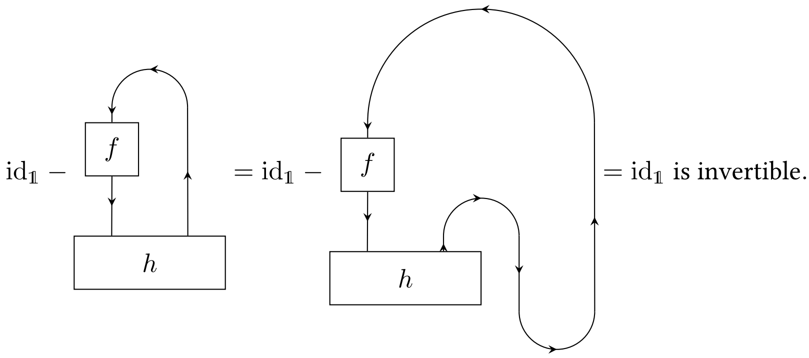
(5.99)

we find that



(5.100)

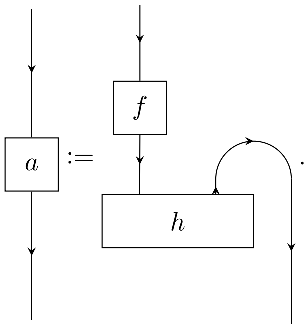
For the other inclusion, suppose that $f : A \to B$ is not invertible. Using Lemma 5.5.2, we know that $g \circ f$ is not invertible for any $g : B \to A^{**}$, and thus that $g \circ f$ is nilpotent through Proposition 5.5.7. Proposition 5.5.8 then implies that $\mathrm{tr}^{\mathrm{left}}(g \circ f) = 0$. Let $h : B \otimes A^* \to \mathbb{1}$ be any morphism, then



(5.101)

We conclude that $\widehat{f} \in \mathrm{rad}(\mathcal{C})(\mathbb{1}, B \otimes A^*) = \mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(\mathbb{1}, B \otimes A^*)$, and thus that $f \in \mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A, B)$.

Suppose now that A is such that for any $a : A \to A^{**}$ (5.100) holds. For any $h : B \otimes A^* \to \mathbb{1}$, we then find that (5.98) holds by setting



(5.102)

Once again, this implies that $\widehat{f} \in \mathrm{rad}(\mathcal{C})(\mathbb{1}, B \otimes A^*) = \mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(\mathbb{1}, B \otimes A^*)$, and thus that $f \in \mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A, B)$. ■

Corollary 5.5.11. *Let $\mathcal{C}$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms, and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be objects of finite length with (unique due to the Krull-Schmidt theorem 2.4.4) decompositions into indecomposable objects $A = \bigoplus_k A_k$ and $B = \bigoplus_\ell B_\ell$. For morphisms $f : A \to B$, this* *implies that we have decompositions* $f = \bigoplus_{k,\ell} f_{k\ell}$ *with* $f_{k\ell} = \mathrm{proj}_{B_\ell} \circ f \circ \mathrm{inc}_{A_k} : A_k \to B_\ell$. *We have*

$$\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A,B) = \left\{ f : A \to B \,\middle|\, (\forall k,\ell) \left(f_{k\ell} \text{ not invertible or } (\forall a : A_k \to A_k{}^{**}) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{left}}(a) \text{ invertible} \right) \right) \right\} \tag{5.103}$$

$$= \left\{ f : A \to B \,\middle|\, (\forall k,\ell) \left(f_{k\ell} \text{ not invertible or } (\forall b : B_\ell \to B_\ell{}^{**}) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{left}}(b) \text{ invertible} \right) \right) \right\}, \tag{5.104}$$

$$\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes}(A,B) = \left\{ f : A \to B \,\middle|\, (\forall k,\ell) \left(f_{k\ell} \text{ not invertible or } (\forall a : A_k \to {}^{**}A_k) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{right}}(a) \text{ invertible} \right) \right) \right\} \tag{5.105}$$

$$= \left\{ f : A \to B \,\middle|\, (\forall k,\ell) \left(f_{k\ell} \text{ not invertible or } (\forall b : B_\ell \to {}^{**}B_\ell) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{right}}(b) \text{ invertible} \right) \right) \right\}. \tag{5.106}$$

Proof. This follows from Proposition 5.2.5 and Proposition 5.5.10. ■

5.6 The structure of the quotient of a tensor category over its maximal tensor ideal

We are now finally ready to finish what we started in Proposition 5.3.7; we will show that the quotient of $\mathcal{C}$ by the maximal tensor ideal is semisimple.

Theorem 5.6.1. *Let* $\mathcal{C}$ *be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms and such that every object has a decomposition into indecomposable objects. Let* $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes} \leq \mathcal{C}$ *and* $\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes} \leq \mathcal{C}$ *be the maximal right and let tensor ideals respectively. The quotient categories* $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ *and* $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes}$ *are semisimple and Schur (and thus abelian through Proposition 5.2.21).*

The non-zero simple objects in $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ *or* $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes}$ *are the images of the objects* $A \in \mathrm{Ob}(\mathcal{C})$ *such that the following do not hold*

$$(\forall a : A \to A^{**}) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{left}}(a) \text{ invertible} \right) \text{ or } (\forall a : A \to {}^{**}A) \left(\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{right}}(a) \text{ invertible} \right), \tag{5.107}$$

or equivalently when $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathbb{1})$ *is a field*

$$(\forall a : A \to A^{**}) \left(\mathrm{tr}^{\mathrm{left}}(a) = 0 \right) \text{ or } (\forall a : A \to {}^{**}A) \left(\mathrm{tr}^{\mathrm{right}}(a) = 0 \right). \tag{5.108}$$

Proof. Proposition 5.2.6 shows that the indecomposables of $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ are the images of the indecomposables in $\mathcal{C}$ under the canonical quotient functor. We will now show that these are simple objects.

Let $A \in \mathrm{Ob}(\mathcal{C})$ be an indecomposable object. If the first condition in (5.107) holds, then we see that $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(B,A) = \mathrm{Hom}_{\mathcal{C}}(B,A)$ for all $B \in \mathrm{Ob}(\mathcal{C})$ (Corollary 5.5.11). We conclude that $\mathrm{quot}_{\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}(A)$ is a null object. If that condition does not hold, then Proposition 5.5.10 shows that $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A,A) \neq \mathrm{Hom}_{\mathcal{C}}(A,A)$ as $\mathrm{id}_A \notin \mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A,A)$. This implies that $\mathrm{quot}_{\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}(A)$ is not a null object.

Suppose now that $A, B \in \mathrm{Ob}(\mathcal{C})$ are two indecomposable objects such that $\mathrm{quot}_{\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}(A), \mathrm{quot}_{\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}(B)$ are not null objects. Proposition 5.5.10 then shows that $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}(A,B)$ consists of the non-invertible morphisms $f : A \to B$. This shows that $\mathrm{Hom}_{\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}\left(\mathrm{quot}_{\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}(A), \mathrm{quot}_{\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}}(B)\right)$ is zero when $A \ncong B$, and a division ring if $A \cong B$. We conclude that $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ is Schur.

Let $X \in \mathrm{Ob}\left(\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}\right)$ be an indecomposable object, and suppose that it has a subobject Y. Without loss of generality we may assume that Y is indecomposable too. Due to the above, we find that the monomorphism $f : Y \to X$ is either zero or an isomorphism, and thus that Y is a null object or isomorphic to X. We conclude that X is simple, and thus that $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ is semisimple.

We will now show that (5.108) is equivalent to (5.107) when $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field. If $\zeta := \mathrm{tr}^{\mathrm{left}}(a) = \mathrm{ev}_{A^*} \circ (a \otimes \mathrm{id}_{A^*}) \circ \mathrm{coev}_A \neq 0$, then ζ is an isomorphism. Through Proposition 4.1.2, we then find

$$\begin{aligned} \mathrm{tr}^{\mathrm{left}}(\zeta^{-1} \cdot a) &= \mathrm{ev}_{A^*} \circ ((\zeta^{-1} \cdot a) \otimes \mathrm{id}_{A^*}) \circ \mathrm{coev}_A \\ &= \mathrm{ev}_{A^*} \circ (\zeta^{-1} \cdot (a \otimes \mathrm{id}_{A^*})) \circ \mathrm{coev}_A \\ &= \zeta^{-1} \cdot (\mathrm{ev}_{A^*} \circ (a \otimes \mathrm{id}_{A^*}) \circ \mathrm{coev}_A) \\ &= \mathrm{id}_{\mathbb{1}} \end{aligned}. \tag{5.109}$$

This is in contradiction with the fact that $\mathrm{id}_{\mathbb{1}} - \mathrm{tr}^{\mathrm{left}}(\zeta^{-1} \cdot a)$ must be invertible. ■

Remark 5.6.2. Note that at no point in the above proof did we assume that the monoidal product is bilinear with respect to the left or right action on the hom-sets defined by $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$.

Remark 5.6.3. Theorem 5.6.1 shows that we have semisimplifications in the sense of Definition 5.0.2, with $\mathbf{Struct}$ = abelian rigid monoidal categories such that the monoidal product is bilinear and such that all objects have decompositions into indecomposable objects, $\mathcal{M} = \mathbf{Ab}$, and $\mathrm{BinOp} = \{\circ, \otimes\}$. In particular, this works for all multitensor categories over rings in which every object has a decomposition into indecomposable objects.

Indeed, if $\mathcal{I} \leq \mathcal{C}$ is a right or left tensor ideal such that $\mathcal{C}/\mathcal{I}$ is semisimple and Schur, then $\mathrm{rad}(\mathcal{C}) \subseteq \mathcal{I}$ due to Corollary 5.2.18. Remark 5.3.6 then implies that $\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes} \subseteq \mathcal{I}$ or $\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes} \subseteq \mathcal{I}$.

Furthermore, if $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a field (e.g. when $\mathcal{C}$ is a tensor category), then Proposition 5.3.7 implies that the only non-zero quotient of $\mathcal{C}$ over a right or left tensor ideal is exactly $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{left}}_{\otimes}$ or $\mathcal{C}/\mathrm{rad}(\mathcal{C})^{\mathrm{right}}_{\otimes}$.

6 Algebras in Monoidal Categories

This chapter is devoted to the study of algebras in monoidal and symmetric tensor categories. We begin with a general treatment in the setting of monoidal categories and gradually introduce additional structures, allowing us to develop more intricate concepts.

While most of the existing literature focuses on unital associative algebras (see, for example, [EGNO15; Cou23a; Cou23b]), our approach is fully general: we make no assumptions of associativity or the existence of a unit. This broader perspective includes classical examples but also accommodates more exotic structures that arise naturally in settings such as the Verlinde category, where non-associative algebras can play a central role.

A new contribution of this chapter is a construction of the ideal generated by a subobject in a non-associative algebra. To the best of our knowledge, this construction does not appear in the existing literature.

6.1 Magmas and monoids in monoidal categories

6.1.1 Magmas and monoids

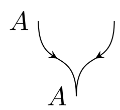
Over a commutative ring R, an algebra is a pair (A, μ), where A is an R-module and $\mu : A \times A \to A$ is a bilinear map. Equivalently, by the universal property of the tensor product of modules, this is a morphism $\mu : A \otimes_R A \to A$ in the category ${}_R\mathbf{Mod}$.

This notion admits a natural generalisation to arbitrary monoidal categories. In monoidal categories that are not pre-additive, we will refer to such structures not as algebras, but as magmas.

Definition 6.1.1 ((Co)magmas and (co)monoids, [EGNO15, Definition 7.8.1]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. A *magma* in $\mathcal{C}$ is a pair (A, μ) of an object $A \in \mathrm{Ob}(\mathcal{C})$, and a morphism $\mu : A \otimes A \to A$, called the *multiplication*.

In the graphical calculus of string diagrams, we draw multiplications as

$\mu =$



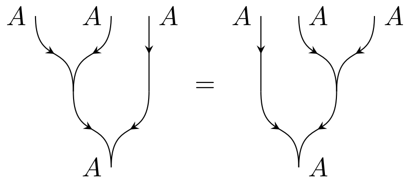
. (6.1)

A magma (A, μ) is called

1. *associative* if the following diagram commutes

$$\begin{array}{ccc} (A \otimes A) \otimes A & \xrightarrow{\mu \otimes \mathrm{id}_A} & A \otimes A \\ \downarrow \alpha_{(A,A,A)} & & \searrow \mu \\ & & A \\ & & \nearrow \mu \\ A \otimes (A \otimes A) & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes A \end{array}, \tag{6.2}$$

or graphically

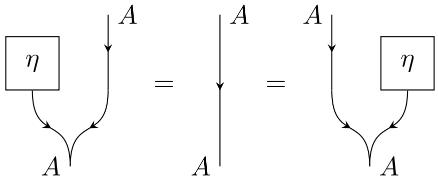


$$\quad, \tag{6.3}$$

2. *unital* if there exists a morphism $\eta : \mathbb{1} \to A$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{1} \otimes A & \xrightarrow{\eta \otimes \mathrm{id}_A} & A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \eta} & A \otimes \mathbb{1} \\ & \searrow^{\lambda_A} & \downarrow \mu & \swarrow_{\rho_A} & \\ & & A & & \end{array} \quad, \tag{6.4}$$

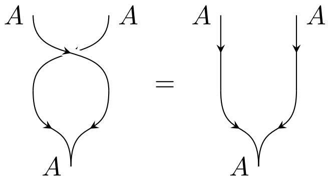
or graphically



$$, \tag{6.5}$$

note that this implies that μ is a split epimorphism,

3. *commutative* (or *anti-commutative*[1]) if, in addition, $\mathcal{C}$ is equipped with a braiding γ, and $\mu \circ \gamma_{(A,A)} = \mu$ (or $\mu \circ \gamma_{(A,A)} = -\mu$). Graphically, this is

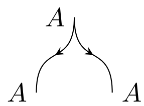


$$. \tag{6.6}$$

A unital and associative magma is called a *monoid*.

Dually, a *comagma* is a pair (A, μ) of an object $A \in \mathrm{Ob}(\mathcal{C})$, and a morphism $\mu : A \to A \otimes A$, called the *comultiplication*.

Graphically, we draw comultiplications as

$\mu =$ 

$$. \tag{6.7}$$

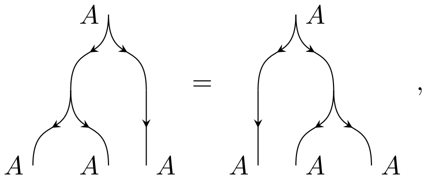
A comagma (A, μ) is called

1. *coassociative* if the following diagram commutes

$$\begin{array}{ccc} & A \otimes A \xrightarrow{\mu \otimes \mathrm{id}_A} (A \otimes A) \otimes A & \\ {}^{\mu}\nearrow & & \\ A & & \downarrow \alpha_{(A,A,A)} \\ {}_{\mu}\searrow & & \\ & A \otimes A \xrightarrow{\mathrm{id}_A \otimes \mu} A \otimes (A \otimes A) & \end{array} \quad, \tag{6.8}$$

[1]The category needs to be pre-additive for this to make sense.

or graphically

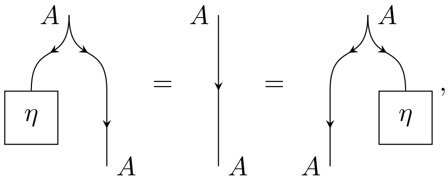


(6.9)

2. *counital* if there exists a morphism $\eta : A \to \mathbb{1}$ such that the following diagram commutes

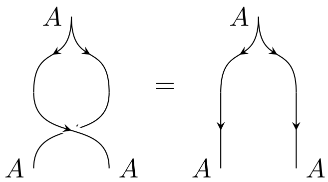
$$\begin{array}{ccccc} & \overset{\lambda_A}{\nearrow} & A & \overset{\rho_A}{\nwarrow} & \\ & & \downarrow \mu & & \\ \mathbb{1} \otimes A & \xleftarrow{\eta \otimes \mathrm{id}_A} & A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \eta} & A \otimes \mathbb{1} \end{array}, \tag{6.10}$$

or graphically



(6.11)

3. *cocommutative* (or *anti-cocommutative*) if, in addition, $\mathcal{C}$ is equipped with a braiding γ, and $\gamma_{(A,A)} \circ \mu = \mu$ (or $\gamma_{(A,A)} \circ \mu = -\mu$). Graphically, this is



(6.12)

A counital and coassociative comagma is called a *comonoid*.

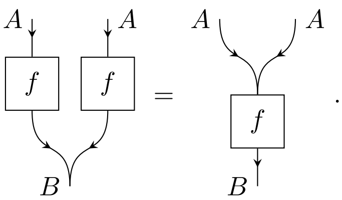
Example 41. In the strict monoidal category $(\mathbf{Set}, \times, \{\star\})$ equipped with the swap map (associative, unital, commutative, anti-commutative) magmas are (associative, unital, commutative, anti-commutative) magmas in the usual sense.

Definition 6.1.2 (Morphisms between (co)magmas and (co)monoids). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. We have the following notions of morphisms between magmas.

1. Let (A, μ) and (B, ν) be two magmas in $\mathcal{C}$. A *magma morphism* $(A, \mu) \to (B, \nu)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ \mu \downarrow & & \downarrow \nu \\ A & \xrightarrow{f} & B \end{array}. \tag{6.13}$$

Graphically, this is



(6.14)

2. Let (A, μ, η) and (B, ν, θ) be two unital magmas in $\mathcal{C}$. A *unital magma morphism* $(A, \mu, \eta) \to (B, \nu, \theta)$ is a magma morphism $f : (A, \mu) \to (B, \nu)$ such that $f \circ \eta = \theta$.

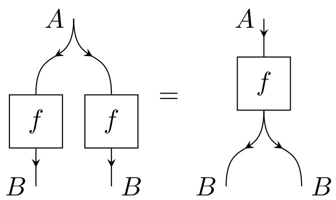
These morphisms define categories $\mathbf{Magma}_{\mathcal{C}}$ and $\mathbf{UnitMagma}_{\mathcal{C}}$ of all magmas and all unital magmas in $\mathcal{C}$. We can then define categories of associative, commutative, anti-commutative, … magmas as full subcategories of $\mathbf{Magma}_{\mathcal{C}}$. Similarly we define categories of unital associative, commutative, anti-commutative, … magmas as full subcategories of $\mathbf{UnitMagma}_{\mathcal{C}}$. In particular, $\mathbf{AssCommUnitMagma}_{\mathcal{C}}$ is the category of commutative monoids in $\mathcal{C}$.

Dually, we have the following notions of morphisms between comagmas.

1. Let (A, μ) and (B, ν) be two comagmas in $\mathcal{C}$. A *comagma morphism* $(A, \mu) \to (B, \nu)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ A \otimes A & \xrightarrow{f\otimes f} & B \otimes B \end{array} \quad . \tag{6.15}$$

Graphically, this is



$$. \tag{6.16}$$

2. Let (A, μ, η) and (B, ν, θ) be two counital comagmas in $\mathcal{C}$. A *counital comagma morphism* $(A, \mu, \eta) \to (B, \nu, \theta)$ is a comagma morphism $f : (A, \mu) \to (B, \nu)$ such that $\theta \circ f = \eta$.

These morphisms define categories of coassociative, cocommutative, anti-cocommutative, … comagmas as full subcategories of the category of comagmas $\mathbf{Comagma}_{\mathcal{C}}$, and categories of counital coassociative, cocommutative, anti-cocommutative, … comagmas as full subcategories of the category of counital comagmas $\mathbf{CounitComagma}_{\mathcal{C}}$.

Example 42 (Monoidal product of magmas). Suppose that, in addition, $\mathcal{C}$ is equipped with a braiding γ. Provided with two magmas (A, μ) and (B, ν) in $\mathcal{C}$, we can define the *monoidal product magma* $(A, \mu) \otimes (B, \nu) = (A \otimes B, \mu\nu)$ through

$$\mu\nu = (\mu \otimes \nu) \circ (\mathrm{id}_A \otimes \gamma_{(B,A)} \otimes \mathrm{id}_B). \tag{6.17}$$

Suppose that we have magma morphisms $\sigma : (A_1, \mu_1) \to (A_2, \mu_2)$ and $\tau : (B_1, \nu_1) \to (B_2, \nu_2)$. The monoidal product $\sigma \otimes \tau$ in $\mathcal{C}$ then becomes a morphism $(A_1, \mu_1) \otimes (B_1, \nu_1) \to (A_2, \mu_2) \otimes (B_2, \nu_2)$, as

$$\begin{aligned} (\sigma \otimes \tau) \circ (\mu_1 \otimes \nu_1) \circ (\mathrm{id}_{A_1} \otimes \gamma_{(B_1,A_1)} \otimes \mathrm{id}_{B_1}) &= (\mu_2 \otimes \nu_2) \circ (\sigma \otimes \sigma \otimes \tau \otimes \tau) \circ (\mathrm{id}_{A_1} \otimes \gamma_{(B_1,A_1)} \otimes \mathrm{id}_{B_1}) \\ &= (\mu_2 \otimes \nu_2) \circ (\mathrm{id}_{A_2} \otimes \gamma_{(B_2,A_2)} \otimes \mathrm{id}_{B_2}) \circ (\sigma \otimes \tau \otimes \sigma \otimes \tau) \end{aligned} . \tag{6.18}$$

6.1.2 Actions and modules of magmas

We can also generalise the notions of modules over algebras in the monoidal setting.

Definition 6.1.3 ((Co)actions of (co)magmas on objects, [EGNO15, Definition 7.8.5]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category, and let $X \in \mathrm{Ob}(\mathcal{C})$.

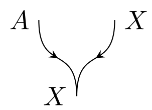
1. Let (A, μ) be an associative magma.

a) A *left action* of (A,μ) on X is a morphism $\triangleright : A\otimes X \to X$ such that the following diagram commutes

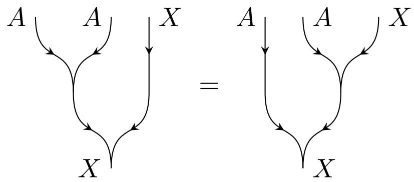
$$\begin{array}{ccc} (A\otimes A)\otimes X & \xrightarrow{\mu\otimes \mathrm{id}_X} & A\otimes X \\ \downarrow{\scriptstyle \alpha_{(A,A,X)}} & & \searrow{\scriptstyle\triangleright} \\ & & X \\ & & \nearrow{\scriptstyle\triangleright} \\ A\otimes (A\otimes X) & \xrightarrow{\mathrm{id}_A\otimes\triangleright} & A\otimes X \end{array} . \tag{6.19}$$

The pair $(X,\triangleright)$ is called a *left module* over (A,μ).

Graphically, we draw

$\triangleright =$  , (6.20)

and the compatibility constraint (6.19) then becomes

 . (6.21)

b) A *right action* of (A,μ) on X is a morphism $\triangleleft : X\otimes A \to X$ such that the following diagram commutes

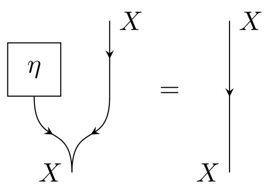
$$\begin{array}{ccc} X\otimes (A\otimes A) & \xrightarrow{\mathrm{id}_X\otimes\mu} & X\otimes A \\ \uparrow{\scriptstyle \alpha_{(X,A,A)}} & & \searrow{\scriptstyle\triangleleft} \\ & & X \\ & & \nearrow{\scriptstyle\triangleleft} \\ (X\otimes A)\otimes A & \xrightarrow{\triangleleft\otimes\mathrm{id}_A} & X\otimes A \end{array} . \tag{6.22}$$

The pair $(X,\triangleleft)$ is called a *right module* over (A,μ).

The graphical language for right actions is the same as for left actions after reflecting over the vertical axis.

2. Let (A,μ,η) be an associative unital magma.

a) A left action of (A,μ,η) on X is a left action $\triangleright : A\otimes X\to X$ of (A,μ) on X such that $\triangleright\circ(\eta\otimes \mathrm{id}_X) = \lambda_X$. Graphically, this is

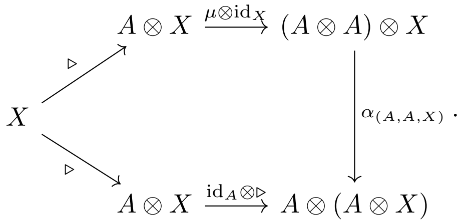
 . (6.23)

b) A right action of (A,μ,η) on X is a right action $\triangleleft : X\otimes A\to X$ of (A,μ) on X such that $\triangleleft\circ(\mathrm{id}_X\otimes\eta) = \rho_X$.

Dually, we have the following notions.

1. Let (A,μ) be a coassociative comagma.

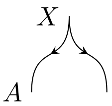
a) A *left coaction* of (A, μ) on X is a morphism $\triangleright : X \to A \otimes X$ such that the following diagram commutes



(6.24)

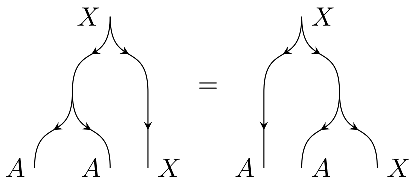
The pair $(X, \triangleright)$ is called a *left comodule* over (A, μ).

Graphically, we draw

$\triangleright =$ 

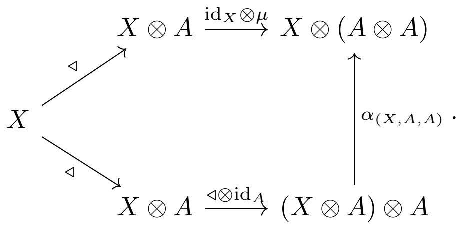
, (6.25)

and the compatibility constraint (6.24) then becomes



. (6.26)

b) A *right coaction* of (A, μ) on X is a morphism $\triangleleft : X \to X \otimes A$ such that the following diagram commutes



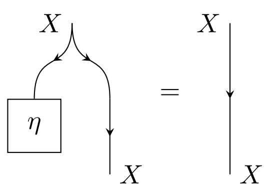
(6.27)

The pair $(X, \triangleleft)$ is called a *right comodule* over (A, μ).

The graphical language for right coactions is the same as for left coactions after reflecting over the vertical axis.

2. Let (A, μ, η) be a coassociative counital comagma.

a) A left coaction of (A, μ, η) on X is a left coaction $\triangleright : X \to A \otimes X$ of (A, μ) on X such that $(\eta \otimes \mathrm{id}_X) \circ \triangleright = \lambda_X^{-1}$. Graphically, this is



. (6.28)

b) A right coaction of (A, μ, η) on X is a right coaction $\triangleleft : X \to X \otimes A$ of (A, μ) on X such that $(\mathrm{id}_X \otimes \eta) \circ \triangleleft = \rho_X^{-1}$.

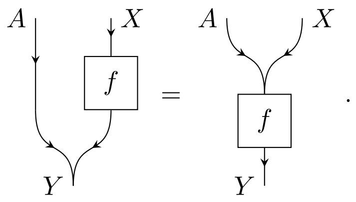
Definition 6.1.4 (Morphisms between (co)modules). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category.

1. Let (A, μ) be an associative magma, possibly unital.

a) Let $(X, \triangleright_X), (Y, \triangleright_Y)$ be two left modules over (A, μ). A *module morphism* $(X, \triangleright_X) \to (Y, \triangleright_Y)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} A \otimes X & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes Y \\ {\scriptstyle \triangleright_X}\downarrow & & \downarrow{\scriptstyle \triangleright_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} . \tag{6.29}$$

Graphically, this is



(6.30)

b) Let $(X, \triangleleft_X), (Y, \triangleleft_Y)$ be two right modules over (A, μ). A module morphism $(X, \triangleleft_X) \to (Y, \triangleleft_Y)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} X \otimes A & \xrightarrow{f \otimes \mathrm{id}_A} & Y \otimes A \\ {\scriptstyle \triangleleft_X}\downarrow & & \downarrow{\scriptstyle \triangleleft_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} . \tag{6.31}$$

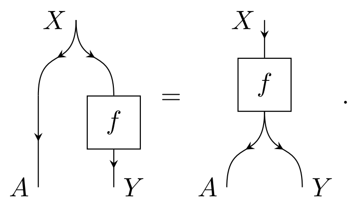
These morphisms define categories ${}_{(A,\mu)}\mathbf{Mod}$ and $\mathbf{Mod}_{(A,\mu)}$ of left and right modules respectively.

2. Let (A, μ) be a coassociative comagma, possibly counital.

a) Let $(X, \triangleright_X), (Y, \triangleright_Y)$ be two left comodules over (A, μ). A *comodule morphism* $(X, \triangleright_X) \to (Y, \triangleright_Y)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \triangleright_X}\downarrow & & \downarrow{\scriptstyle \triangleright_Y} \\ A \otimes X & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes Y \end{array} . \tag{6.32}$$

Graphically, this is



(6.33)

b) Let $(X, \triangleleft_X), (Y, \triangleleft_Y)$ be two right comodules over (A, μ). A comodule morphism $(X, \triangleleft_X) \to (Y, \triangleleft_Y)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \triangleleft_X}\downarrow & & \downarrow{\scriptstyle \triangleleft_Y} \\ X \otimes A & \xrightarrow{f \otimes \mathrm{id}_A} & Y \otimes A \end{array} . \tag{6.34}$$

These morphisms define categories ${}_{(A,\mu)}\mathbf{Comod}$ and $\mathbf{Comod}_{(A,\mu)}$ of left and right comodules respectively.

6.1.3 Ideals in magmas

Finally, we generalise ideals in algebras to the monoidal setting.

Definition 6.1.5 ((Co)ideals in (co)magmas, [Ven23, Definition 2.7.1]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category.

1. Let (A, μ) be a magma in $\mathcal{C}$.

   a) A *left ideal* in (A, μ) is a pair (I, i) of an object $I \in \mathrm{Ob}(\mathcal{C})$, together with a monomorphism $i : I \to A$ which induces a (necessarily unique, as i is a monomorphism) morphism $\mu_I : A \otimes I \to I$ making the following diagram commute

   $$\begin{tikzcd} A \otimes I \arrow[r, "\mathrm{id}_A \otimes i"] \arrow[dr, dashed, red, "\exists! \mu_I"'] & A \otimes A \arrow[r, "\mu"] & A \\ & I \arrow[ur, "i"'] & \end{tikzcd} . \tag{6.35}$$

   b) A *right ideal* in (A, μ) is a pair (I, i) of an object $I \in \mathrm{Ob}(\mathcal{C})$, together with a monomorphism $i : I \to A$ which induces a (necessarily unique) morphism $\mu_I : A \otimes I \to I$ making the following diagram commute

   $$\begin{tikzcd} I \otimes A \arrow[r, "i \otimes \mathrm{id}_A"] \arrow[dr, dashed, red, "\exists! \mu_I"'] & A \otimes A \arrow[r, "\mu"] & A \\ & I \arrow[ur, "i"'] & \end{tikzcd} . \tag{6.36}$$

   c) A *(double) ideal* in (A, μ) is a pair (I, i) of an object $I \in \mathrm{Ob}(\mathcal{C})$, together with a monomorphism $i : I \to A$, which is both a left and a right ideal.

2. Let (A, μ) be a comagma in $\mathcal{C}$.

   a) A *left coideal* in (A, μ) is a pair (Q, q) of an object $Q \in \mathrm{Ob}(\mathcal{C})$, together with an epimorphism $q : A \to Q$ which induces a (necessarily unique) morphism $\mu_Q : Q \to A \otimes Q$ making the following diagram commute

   $$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[dr, "q"'] & A \otimes A \arrow[r, "\mathrm{id}_A \otimes q"] & A \otimes Q \\ & Q \arrow[ur, dashed, red, "\exists! \mu_Q"'] & \end{tikzcd} . \tag{6.37}$$

   b) A *right coideal* in (A, μ) is a pair (Q, q) of an object $Q \in \mathrm{Ob}(\mathcal{C})$, together with an epimorphism $q : A \to Q$ which induces a (necessarily unique) morphism $\mu_Q : Q \to Q \otimes A$ making the following diagram commute

   $$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[dr, "q"'] & A \otimes A \arrow[r, "q \otimes \mathrm{id}_A"] & Q \otimes A \\ & Q \arrow[ur, dashed, red, "\exists! \mu_Q"'] & \end{tikzcd} . \tag{6.38}$$

   c) A *(double) coideal* in (A, μ) is a pair (Q, q) of an object $Q \in \mathrm{Ob}(\mathcal{C})$, together with an epimorphism $q : Q \to A$, which is both a left and a right coideal.

In associative magmas, ideals should be “modules which are contained in the magma”.

Proposition 6.1.6. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category.*

1. *Let (A, μ) be a (possibly unital) magma. A left or right ideal (I, i) in (A, μ) is a left or right module over (A, μ).*

2. *Let (A, μ) be a (possibly counital) comagma. A left or right coideal (I, i) in (A, μ) is a left or right comodule over (A, μ).*

Proof. Suppose first that (A, μ) is not unital. We will prove that a left ideal (I, i) in (A, μ) is a left module over (A, μ), when equipped with the left action $\triangleright = \mu_I$.

We have to prove that (6.19) commutes. We have

$$\begin{aligned} i \circ \triangleright \circ (\mu \otimes \mathrm{id}_I) &= \mu \circ (\mathrm{id}_A \otimes i) \circ (\mu \otimes \mathrm{id}_I) \\ &= \mu \circ (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A\otimes A} \otimes i) \\ &= \mu \circ (\mathrm{id}_A \otimes \mu) \circ \alpha_{(A,A,A)} \circ (\mathrm{id}_{A\otimes A} \otimes i) \\ &= \mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \otimes i)) \circ \alpha_{(A,A,I)}, \\ &= \mu \circ (\mathrm{id}_A \otimes (i \circ \mu_I)) \circ \alpha_{(A,A,I)} \\ &= i \circ \mu_I \circ (\mathrm{id}_A \otimes \mu_I) \circ \alpha_{(A,A,I)} \\ &= i \circ \triangleright \circ (\mathrm{id}_A \otimes \triangleright) \circ \alpha_{(A,A,I)} \end{aligned} \tag{6.39}$$

where we have used that the associator is a natural transformation (3.1), and the ideal condition (6.35). As i is a monomorphism, this shows that $(I, \triangleright)$ is a left module over (A, μ).

Suppose now that, in addition, (A, μ) is equipped with a unit $\eta : \mathbb{1} \to A$. We find

$$\begin{aligned} i \circ \triangleright \circ (\eta \otimes \mathrm{id}_A) &= \mu \circ (\mathrm{id}_A \otimes i) \circ (\eta \otimes \mathrm{id}_I) \\ &= \mu \circ (\eta \otimes \mathrm{id}_A) \circ (\mathrm{id}_{\mathbb{1}} \otimes i) \\ &= \lambda_A \circ (\mathrm{id}_{\mathbb{1}} \otimes i) \\ &= i \circ \lambda_I \end{aligned}, \tag{6.40}$$

where we have used the definition of unital magmas (6.4), and the fact that the left unitor is a natural transformation (3.2). We conclude that $(I, \triangleright)$ is also a left unital module if (A, μ) is unital. ∎

We can also show that any ideal of a unital algebra that contains the monoidal unit must coincide with the entire magma.

Proposition 6.1.7. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category.*

1. *Let (A, μ, η) be a unital magma in $\mathcal{C}$, and let (I, i) be an ideal in (A, μ). If the unit $\eta : \mathbb{1} \to A$ factors through the ideal as $\eta = i \circ \eta_I$, then i is an isomorphism and (I, μ_I, η_I) is thus isomorphic as a unital algebra to (A, μ, η).*

2. *Let (A, μ, η) be a counital comagma in $\mathcal{C}$, and let (Q, q) be a coideal in (A, μ). If the counit $\eta : A \to \mathbb{1}$ factors through the coideal as $\eta = \eta_Q \circ q$, then q is an isomorphism and (Q, μ_Q, η_Q) is thus isomorphic as a counital coalgebra to (A, μ, η).*

Proof. If η factors through i as $\eta = i \circ \eta_I$, then

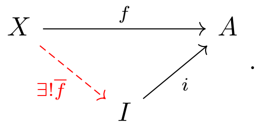
$$\begin{aligned} \mathrm{id}_A &= \mu \circ (\mathrm{id}_A \otimes \eta) \circ \rho_A^{-1} \\ &= \mu \circ (\mathrm{id}_A \otimes i) \circ (\mathrm{id}_A \otimes \eta_I) \circ \rho_A^{-1}. \end{aligned} \tag{6.41}$$

This implies that $\mu \circ (\mathrm{id}_A \otimes i) = i \circ \mu_I$ is a split epimorphism, hence that i is a split epimorphism. As i is also a monomorphism, we conclude that i is an isomorphism. ∎

An important ingredient in many constructions is the notion of the ideal generated by a subset of an algebra. This concept is important in defining finitely generated algebras, as well as in the construction of algebras such as the symmetric and exterior algebras on a set, the universal enveloping algebra of a Lie algebra, and others.

Definition 6.1.8 ((Co)ideals generated by objects). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category.

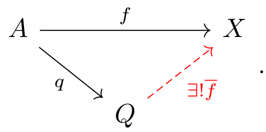
1. Let (A, μ) be a magma in $\mathcal{C}$, and let (X, f) be a subobject of A. A left, right, or double ideal (I, i) is said to *contain* (X, f) if f factors through i, i.e. if there is exists a (necessarily unique) morphism $\overline{f} : X \to I$ making the following diagram commute



. (6.42)

The left, right, double *ideal generated by* (X, f), denoted $(I^{\text{left}}(X, f), i^{\text{left}}(X, f))$, $(I^{\text{right}}(X, f), i^{\text{right}}(X, f))$, $(I(X, f), i(X, f))$, is the smallest left, right, double ideal containing (X, f). This means that for any other ideal (J, j) containing (X, f), this minimal ideal is contained in (J, j).

2. Let (A, μ) be a comagma in $\mathcal{C}$, and let (X, f) be a quotient of A. (X, f) is said to be *a quotient of* a left, right, or double coideal (Q, q) if f factors through q, i.e. if there is exists a (necessarily unique) morphism $\overline{f} : Q \to X$ making the following diagram commute



. (6.43)

The left, right, double *coideal generated by* (X, f), denoted $(Q^{\text{left}}(X, f), q^{\text{left}}(X, f))$, $(Q^{\text{right}}(X, f), q^{\text{right}}(X, f))$, $(Q(X, f), q(X, f))$, is the smallest left, right, double coideal that has (X, f) as a quotient. This means that for any other coideal (R, r) that has (X, f) as a quotient, this minimal coideal is a quotient of (R, r).

6.2 Algebras in enriched monoidal categories

6.2.1 Examples of algebras in monoidal categories

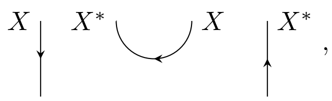
We will now work in monoidal categories which are enriched over some commutative ring, which leads to the proper generalisation of algebras over rings.

Definition 6.2.1 ((Co)algebras). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category that is enriched over some commutative ring R such that the monoidal product is bilinear on morphisms. Magmas in this category are then called *algebras*, and comagmas are called *coalgebras*.

Example 43. Let R be a commutative ring. Algebras in ${}_R\mathbf{Mod}$ are R-algebras, and modules over algebras in this category are modules over R-algebras in the usual sense.

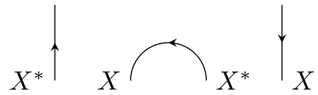
Example 44 (External and internal endomorphism algebras, [EGNO15, Example 7.8.4]). Let $\mathcal{C}$ be monoidal and enriched over a commutative ring R such that the monoidal product is bilinear. For any object $X \in \mathrm{Ob}(\mathcal{C})$, the endomorphism ring $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ is an R-algebra, called the *(external) endomorphism algebra.*

Suppose now that, in addition, $\mathcal{C}$ is left rigid, and set $\underline{\mathrm{End}}^{\text{left}}(X) = \underline{\mathrm{End}}(X) := X \otimes X^*$ and $\overline{\mathrm{End}}^{\text{left}}(X) = \overline{\mathrm{End}}(X) := X^* \otimes X$. We define an algebra $(\underline{\mathrm{End}}(X), \nabla)$ in $\mathcal{C}$ through

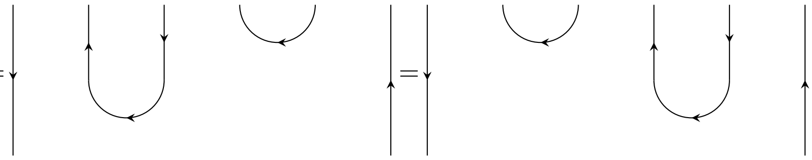
$\nabla =$ 

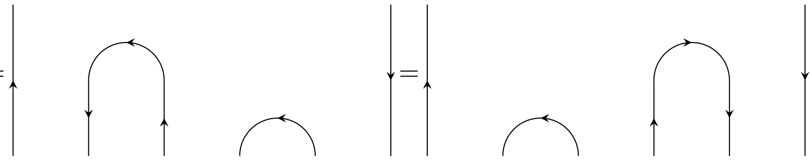
, (6.44)

and a coalgebra $(\overline{\mathrm{End}}(X), \Delta)$ through

$\Delta =$  . (6.45)

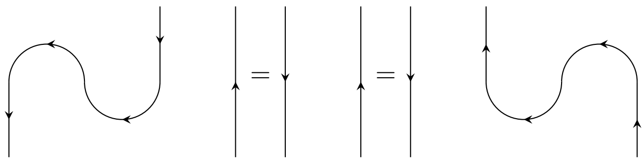
We can prove that $(\underline{\mathrm{End}}(X), \nabla)$ is associative and $(\overline{\mathrm{End}}(X), \Delta)$ coassociative;

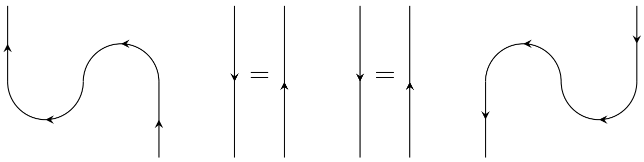
$\nabla \circ (\mathrm{id} \otimes \nabla) =$  $= \nabla \circ (\nabla \otimes \mathrm{id})$,

$(\mathrm{id} \otimes \Delta) \circ \Delta =$  $= (\Delta \otimes \mathrm{id}) \circ \Delta$.

(6.46)

Furthermore, $(\underline{\mathrm{End}}(X), \nabla)$ is unital with unit coev_X, and $(\overline{\mathrm{End}}(X), \Delta)$ is counital with counit ev_X;

$\nabla \circ (\mathrm{coev}_X \otimes \mathrm{id}) =$  $= \nabla \circ (\mathrm{id} \otimes \mathrm{coev}_X)$,

$(\mathrm{ev}_X \otimes \mathrm{id}) \circ \Delta =$  $= (\mathrm{id} \otimes \mathrm{ev}_X) \circ \Delta$.

(6.47)

If $\mathcal{C}$ is right rigid, we similarly have an associative unital algebra structure on $\underline{\mathrm{End}}^{\mathrm{right}}(X) := {}^*X \otimes X$, and a coassociative counital coalgebra structure on $\overline{\mathrm{End}}^{\mathrm{right}}(X) := X \otimes {}^*X$.

Example 45 (External and internal morphism modules, [EGNO15, Example 7.8.11]). Let $\mathcal{C}$ be monoidal and enriched over a commutative ring R such that the monoidal product is bilinear. For any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the hom-space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ equipped with the composition as an action is a right module over $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$, and a left module over $\mathrm{End}_{\mathcal{C}}(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, Y)$.

Similarly, in left rigid categories, $\underline{\mathrm{Hom}}(X, Y) := Y \otimes X^*$ is a right module over $\underline{\mathrm{End}}(X) = X \otimes X^*$, and a left module over $\underline{\mathrm{End}}(Y)$. The actions have a similar structure to the multiplication in Example 44. $\overline{\mathrm{Hom}}(X, Y) := Y^* \otimes X$ is a right comodule over $\overline{\mathrm{End}}(X) = X^* \otimes X$, and a left comodule over $\overline{\mathrm{End}}(Y) = Y^* \otimes Y$.

6.2.2 The unital hull of an algebra

We introduce the unital hull of an algebra, as units sometimes enable more natural or beautiful constructions. This will become apparent, for instance, in our treatment of ideals generated by subobjects.

Definition 6.2.2 ((Co)unital hull of a (co)algebra). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be an additive monoidal category such that the monoidal product is bilinear on morphisms.

1. Let (A, μ) be an algebra in $\mathcal{C}$. The *unital hull* of (A, μ) is the algebra (A_u, μ_u) with $A_u := A \oplus \mathbb{1}$, and

$$\mu_u = \mathrm{inc}_A \circ \mu \circ \mathrm{proj}_{A \otimes A} + \mathrm{inc}_A \circ \rho_A \circ \mathrm{proj}_{A \otimes \mathbb{1}} + \mathrm{inc}_A \circ \lambda_A \circ \mathrm{proj}_{\mathbb{1} \otimes A} + \mathrm{inc}_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \circ \mathrm{proj}_{\mathbb{1} \otimes \mathbb{1}} \,. \quad (6.48)$$

2. Let (A, μ) be a coalgebra in $\mathcal{C}$. The *counital hull* of (A, μ) is the coalgebra (A_u, μ_u) with $A_u := A \oplus \mathbb{1}$, and

$$\mu_u = \mathrm{inc}_{A\otimes A} \circ \mu \circ \mathrm{proj}_A + \mathrm{inc}_{A\otimes \mathbb{1}} \circ \rho_A^{-1} \circ \mathrm{proj}_A + \mathrm{inc}_{\mathbb{1}\otimes A} \circ \lambda_A^{-1} \circ \mathrm{proj}_A + \mathrm{inc}_{\mathbb{1}\otimes\mathbb{1}} \circ \lambda_{\mathbb{1}} \circ \mathrm{proj}_{\mathbb{1}} \,. \tag{6.49}$$

Proposition 6.2.3. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be an additive monoidal category such that the monoidal product is bilinear on morphisms.*

1. *Let (A, μ) be an algebra in $\mathcal{C}$. The unital hull (A_u, μ_u) is unital with unit*

$$\eta_u = \mathrm{inc}_{\mathbb{1}} : \mathbb{1} \to A_u. \tag{6.50}$$

2. *Let (A, μ) be a coalgebra in $\mathcal{C}$. The counital hull (A_u, μ_u) is counital with counit*

$$\eta_u = \mathrm{proj}_{\mathbb{1}} : A_u \to \mathbb{1}. \tag{6.51}$$

Proof. We have

$$\begin{aligned}
\mu_u \circ (\eta_u \otimes \mathrm{id}_{A_u}) &= (\mathrm{inc}_A \circ \lambda_A \circ \mathrm{proj}_{\mathbb{1}\otimes A} + \mathrm{inc}_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \circ \mathrm{proj}_{\mathbb{1}\otimes\mathbb{1}}) \circ (\mathrm{inc}_{\mathbb{1}} \otimes \mathrm{id}_{A_u}) \\
&= (\mathrm{inc}_A \circ \lambda_A \circ \mathrm{proj}_{\mathbb{1}\otimes A} + \mathrm{inc}_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \circ \mathrm{proj}_{\mathbb{1}\otimes\mathbb{1}}) \circ (\mathrm{inc}_{\mathbb{1}} \otimes (\mathrm{inc}_A \circ \mathrm{proj}_A + \mathrm{inc}_{\mathbb{1}} \otimes \mathrm{proj}_A)) \\
&= \mathrm{inc}_A \circ \lambda_A \circ (\mathrm{id}_{\mathbb{1}} \otimes \mathrm{proj}_A) \circ \lambda_{A_u}^{-1} + \mathrm{inc}_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \circ (\mathrm{id}_{\mathbb{1}} \otimes \mathrm{proj}_{\mathbb{1}}) \\
&= \mathrm{inc}_A \circ \mathrm{proj}_A \circ \lambda_{A_u} + \mathrm{inc}_{\mathbb{1}} \circ \mathrm{proj}_{\mathbb{1}} \circ \lambda_{A_u} \\
&= \lambda_{A_u}
\end{aligned} \;, \tag{6.52}$$

and similarly we prove $\mu_u \circ (\mathrm{id}_{A_u} \otimes \eta_u) = \rho_{A_u}$. ∎

6.2.3 Ideals in algebras

An abelian structure allows us to a lot more with ideals in algebras. Most importantly, an abelian structure allows us to define the quotient of an algebra over an ideal.

Quotients of algebras over ideals

Definition 6.2.4 ((Co)quotient of a (co)algebra over a (co)ideal). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be an abelian monoidal category such that the monoidal product is bilinear and biexact.

1. Let (A, μ) be an algebra in $\mathcal{C}$, and let (I, i) be a (double) ideal in (A, μ). Setting $A/I := \mathrm{Coker}(i)$, we have a short exact sequence

$$0 \longrightarrow I \xrightarrow{\;i\;} A \xrightarrow{\mathrm{coker}(i)} A/I \longrightarrow 0 \;. \tag{6.53}$$

Lemma 5.4.1 shows that $\mathrm{coker}(i) \otimes \mathrm{coker}(i)$ is the cokernel of

$$\bar{i} = (i \otimes \mathrm{id}_A) \circ \mathrm{proj}_{I\otimes A} + (\mathrm{id}_A \otimes i) \circ \mathrm{proj}_{A\otimes I} \,. \tag{6.54}$$

As (I, i) is a double ideal, we find

$$\mu \circ \bar{i} = i \circ \mu_I^{\mathrm{right}} \circ \mathrm{proj}_{I\otimes A} + i \circ \mu_I^{\mathrm{left}} \circ \mathrm{proj}_{A\otimes I} \,. \tag{6.55}$$

This implies that $(\mathrm{coker}(i) \circ \mu) \circ \bar{i} = 0$, hence that there exists a uniquely induced morphism $\mu_{A/I} : A/I \otimes A/I \to A/I$ making the following diagram commute

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\;\mu\;} & A \\
{\scriptstyle \mathrm{coker}(i)\otimes \mathrm{coker}(i)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{coker}(i)} \\
A/I \otimes A/I & \overset{\exists! \mu_{A/I}}{\dashrightarrow} & A/I
\end{array} \;. \tag{6.56}$$

The algebra $(A/I, \mu_{A/I})$ is called the *quotient algebra* of (A, μ) over (I, i).

2. Let (A, μ) be a coalgebra in $\mathcal{C}$, and let $(A/I, q)$ with $I = \mathrm{Ker}(q)$ be a (double) coideal over (A, μ). As in the above, there is a unique morphism $\mu_I : I \to I \otimes I$ such that the following diagram commutes

$$\begin{array}{ccc} I & \xrightarrow{\exists ! \mu_I} & I \otimes I \\ {\scriptstyle \ker(q)}\downarrow & & \downarrow{\scriptstyle \ker(q)\otimes\ker(q)} \\ A & \xrightarrow{\mu} & A \otimes A \end{array} . \tag{6.57}$$

The coalgebra (I, μ_I) is called the *coquotient coalgebra* of (A, μ) over (Q, q).

The use of Lemma 5.4.1 shows that it is not entirely trivial that quotients of algebras over ideals are algebras in our general setting.

Ideals in unital algebras

In categories that admit kernels, we will show that we can construct ideals in a non-unital algebra by using the ideals of its unital hull. This is convenient, as working with unital algebras is often simpler.

More generally, we will now introduce a *unital hull functor* on subobjects. Recall the definition of categories of subobjects and quotients, Definition 1.1.6.

Definition 6.2.5. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a pre-abelian monoidal category such that the monoidal product is bilinear on morphisms. Let $A \in \mathrm{Ob}\,(\mathcal{C})$ and let $A_u := A \oplus \mathbb{1}$. We define the following maps on objects

$$\mathrm{UnitHull}_A : \mathrm{Sub}(A) \to \mathrm{Sub}(A_u) : (X, f) \mapsto (X, \mathrm{inc}_A \circ f), \tag{6.58}$$
$$\overline{\mathrm{UnitHull}}_A : \mathrm{Sub}(A_u) \to \mathrm{Sub}(A) : (X_u, f_u) \mapsto (\mathrm{Ker}(\mathrm{proj}_{\mathbb{1}} \circ f_u), \mathrm{proj}_A \circ f_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u)), \tag{6.59}$$

and

$$\mathrm{CounitHull}_A : \mathrm{Quot}(A) \to \mathrm{Quot}(A_u) : (X, f) \mapsto (X, f \circ \mathrm{proj}_A), \tag{6.60}$$
$$\overline{\mathrm{CounitHull}}_A : \mathrm{Quot}(A_u) \to \mathrm{Quot}(A) : (X_u, f_u) \mapsto (\mathrm{Coker}(f_u \circ \mathrm{inc}_{\mathbb{1}}), \mathrm{coker}(f_u \circ \mathrm{inc}_{\mathbb{1}}) \circ f_u \circ \mathrm{inc}_A). \tag{6.61}$$

Lemma 6.2.6. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a pre-abelian monoidal category such that the monoidal product is bilinear on morphisms, and let $A \in \mathrm{Ob}\,(\mathcal{C})$. The above maps* $\mathrm{UnitHull}_A, \overline{\mathrm{UnitHull}}_A, \mathrm{CounitHull}_A, \overline{\mathrm{CounitHull}}_A$ *define functors, and*

$$\overline{\mathrm{UnitHull}}_A \circ \mathrm{UnitHull}_A = \mathrm{id}_{\mathrm{Sub}(A)}, \tag{6.62}$$
$$\overline{\mathrm{CounitHull}}_A \circ \mathrm{CounitHull}_A = \mathrm{id}_{\mathrm{Quot}(A)}. \tag{6.63}$$

Proof. We will first prove that these maps are well-defined. If $(X, f) \in \mathrm{Ob}\,\big(\mathrm{Sub}(A)\big)$, then $\mathrm{inc}_A \circ f$ is a monomorphism, which shows that $\mathrm{UnitHull}_A(X, f)$ is a subobject of A_u. If $(X_u, f_u) \in \mathrm{Ob}\,\big(\mathrm{Sub}(A_u)\big)$, then

$$\begin{aligned} \mathrm{inc}_A \circ \mathrm{proj}_A \circ f_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u) &= (\mathrm{inc}_A \circ \mathrm{proj}_A + \mathrm{inc}_{\mathbb{1}} \circ \mathrm{proj}_{\mathbb{1}}) \circ f_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u) \\ &= f_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u) \end{aligned} \tag{6.64}$$

is a monomorphism, which implies that $\overline{\mathrm{UnitHull}}_A(X_u, f_u)$ is a subobject of A.

Next, we show that these maps induce a map on morphisms. Let $\overline{f} : X \to Y$ be such that $f = g \circ \overline{f}$ for $(X, f), (Y, g) \in \mathrm{Ob}\,\big(\mathrm{Sub}(A)\big)$. We define

$$\mathrm{UnitHull}_A(\overline{f}) = \overline{f}, \tag{6.65}$$

and we trivially have $\mathrm{inc}_A \circ g \circ \overline{f} = \mathrm{inc}_A \circ f$.

Let $\overline{f}_u : X_u \to Y_u$ be such that $f_u = g_u \circ \overline{f}_u$ for $(X_u, f_u), (Y_u, g_u) \in \mathrm{Ob}\left(\mathrm{Sub}(A_u)\right)$. We then find

$$(\mathrm{proj}_{\mathbb{1}} \circ g_u) \circ (\overline{f}_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u)) = (\mathrm{proj}_{\mathbb{1}} \circ f_u) \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u) = 0, \tag{6.66}$$

which implies that there is a uniquely induced morphism $\overline{f} : \mathrm{Ker}(\mathrm{proj}_{\mathbb{1}} \circ f_u) \to \mathrm{Ker}(\mathrm{proj}_{\mathbb{1}} \circ g_u)$ such that $\ker(\mathrm{proj}_{\mathbb{1}} \circ g_u) \circ \overline{f} = \overline{f}_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u)$. Setting

$$\overline{\mathrm{UnitHull}}_A(\overline{f}_u) = \overline{f}, \tag{6.67}$$

we obtain

$$\mathrm{proj}_A \circ g_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ g_u) \circ \overline{f} = \mathrm{proj}_A \circ g_u \circ \overline{f}_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u) = \mathrm{proj}_A \circ f_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ f_u). \tag{6.68}$$

The fact that these maps are functors now follows from the fact that all hom-sets contain at most one morphism.

Finally, $\overline{\mathrm{UnitHull}}_A \circ \mathrm{UnitHull}_A = \mathrm{id}_{\mathrm{Sub}(A)}$ follows from $\mathrm{proj}_{\mathbb{1}} \circ \mathrm{inc}_A \circ f = 0$ and $\mathrm{proj}_A \circ \mathrm{inc}_A \circ f = f$. ■

We will now prove that the unital hull functor preserves ideals of algebras.

Proposition 6.2.7. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a pre-abelian monoidal category such that the monoidal product is bilinear on morphisms.*

1. *Let (A, μ) be an algebra in $\mathcal{C}$, and let (A_u, μ_u, η_u) be the unital hull of this algebra. $\mathrm{UnitHull}_A$ and $\overline{\mathrm{UnitHull}}_A$ restrict to functors on the categories of ideals (which are the full subcategories of $\mathrm{Sub}(A)$ and $\mathrm{Sub}(A_u)$ consisting of all ideals), by which we mean that*
   a) *provided with a left, right, or double ideal (I, i) in (A, μ), $\mathrm{UnitHull}_A(I, i)$ is a left, right, or double ideal in (A_u, μ_u),*
   b) *provided with a left, right, or double ideal (I_u, i_u) in (A_u, μ_u), $\overline{\mathrm{UnitHull}}_A(I_u, i_u)$ is a left, right, or double ideal in (A, μ).*
2. *Let (A, μ) be a coalgebra in $\mathcal{C}$, and let (A_u, μ_u, η_u) be the counital hull of this coalgebra. $\mathrm{CounitHull}_A$ and $\overline{\mathrm{CounitHull}}_A$ restrict to functors on the categories of coideals (which are the full subcategories of $\mathrm{Quot}(A)$ and $\mathrm{Quot}(A_u)$ consisting of all coideals), by which we mean that*
   a) *provided with a left, right, or double coideal (Q, q) over (A, μ), $\mathrm{CounitHull}_A(Q, q)$ is a left, right, or double coideal over (A_u, μ_u),*
   b) *provided with a left, right, or double coideal (Q_u, q_u) over (A_u, μ_u), $\overline{\mathrm{CounitHull}}_A(Q_u, q_u)$ is a left, right, or double coideal over (A, μ).*

Proof. We denote $\mathrm{UnitHull}_A(I, i) = (I_u, i_u)$ and $\overline{\mathrm{UnitHull}}_A(I_u, i_u) = (I, i)$.

Suppose that (I_u, i_u) is a left ideal in (A_u, μ_u). This implies that there exists a morphism $\mu_u^{I_u} : A_u \otimes I_u \to I_u$ such that

$$\mu_u \circ (\mathrm{id}_{A_u} \otimes i_u) = i_u \circ \mu_u^{I_u}. \tag{6.69}$$

We thus find

$$\mathrm{proj}_{\mathbb{1}} \circ i_u \circ \mu_u^{I_u} \circ (\mathrm{inc}_A \otimes \mathrm{id}_{I_u}) = \mathrm{proj}_{\mathbb{1}} \circ \mu_u \circ (\mathrm{inc}_A \otimes i_u) = 0, \tag{6.70}$$

from which we conclude that there is a uniquely induced morphism $\mu_I : A \otimes I \to \mathrm{Ker}(\mathrm{proj}_{\mathbb{1}} \circ i_u)$ such that the following diagram commutes

$$\begin{array}{ccccccc} A \otimes I & \xrightarrow{\mathrm{id}_A \otimes \ker(\mathrm{proj}_{\mathbb{1}} \circ i_u)} & A \otimes I_u & \xrightarrow{\mathrm{inc}_A \otimes \mathrm{id}_{I_u}} & A_u \otimes I_u & \xrightarrow{\mu_u^{I_u}} & I_u \\ {\scriptstyle \exists ! \mu_I} \downarrow & & & & & \nearrow {\scriptstyle \ker(\mathrm{proj}_{\mathbb{1}} \circ i_u)} & \\ \mathrm{Ker}(\mathrm{proj}_{\mathbb{1}} \circ i_u) & & & & & & \end{array} . \tag{6.71}$$

Using this morphism, we see

$$\begin{aligned} i \circ \mu_I &= \mathrm{proj}_A \circ i_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ i_u) \circ \mu_I \\ &= \mathrm{proj}_A \circ i_u \circ \mu_u^{I_u} \circ (\mathrm{inc}_A \otimes \ker(\mathrm{proj}_{\mathbb{1}} \circ i_u)) \\ &= \mathrm{proj}_A \circ \mu_u \circ (\mathrm{inc}_A \otimes (i_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ i_u))) \\ &= (\mu \circ (\mathrm{proj}_A \otimes \mathrm{proj}_A) + \rho_A \circ (\mathrm{proj}_A \otimes \mathrm{proj}_{\mathbb{1}})) \circ (\mathrm{inc}_A \otimes (i_u \circ \ker(\mathrm{proj}_{\mathbb{1}} \circ i_u))) \\ &= \mu \circ (\mathrm{id}_A \otimes i) \end{aligned} . \tag{6.72}$$

We conclude that $(I, i) = \overline{\mathrm{UnitHull}}(I_u, i_u)$ is an ideal in (A, μ) as we already know that i is a monomorphism from Lemma 6.2.6.

Suppose now that (I, i) is a left ideal in (A, μ). We obtain

$$\begin{aligned} \mu_u \circ (\mathrm{id}_{A_u} \otimes i_u) &= \mathrm{inc}_A \circ \mu \circ (\mathrm{proj}_A \otimes i) + \mathrm{inc}_A \circ \lambda_A \circ (\mathrm{proj}_{\mathbb{1}} \otimes i) \\ &= \mathrm{inc}_A \circ i \circ \mu_I \circ (\mathrm{proj}_A \otimes \mathrm{id}_I) + \mathrm{inc}_A \circ i \circ \lambda_I \circ (\mathrm{proj}_{\mathbb{1}} \otimes \mathrm{id}_I), \\ &= i_u \circ (\mu_I \circ (\mathrm{proj}_A \otimes \mathrm{id}_I) + \lambda_I \circ (\mathrm{proj}_{\mathbb{1}} \otimes \mathrm{id}_I)) \end{aligned} \tag{6.73}$$

which implies that $(I_u, i_u) = \mathrm{UnitHull}(I, i)$ is a left ideal in (A_u, μ_u) by setting

$$\mu_u^{I_u} := \mu_I \circ (\mathrm{proj}_A \otimes \mathrm{id}_I) + \lambda_I \circ (\mathrm{proj}_{\mathbb{1}} \otimes \mathrm{id}_I). \tag{6.74}$$

■

Ideals generated by a subobject

Ideals generated by a subset X of an associative unital R-algebra A are very easy to define as $\langle X \rangle = RX = \{r \cdot x \mid r \in R,\, x \in X\}$ (for left ideals). In non-associative algebras this becomes a bit harder: $r \cdot x$ must be included in $\langle X \rangle$ for all $r \in R$ and $x \in X$, but also $s \cdot (r \cdot x)$, which may not equal $(s \cdot r) \cdot x$, must be included in $\langle X \rangle$ for all $r, s \in R$ and $x \in X$.

More generally, we obtain a sequence of sets $X, RX, R(RX), R(R(RX)), \ldots$ each included in the next, and all of which should be contained in the ideal generated by X. It is then clear that the colimit of this sequence must be included in the ideal generated by X, and it is not very hard to convince yourself that this colimit is in fact equal to the ideal generated by X (as it is $\bigcup_n R^n X$).

This is a procedure we can generalise in abelian monoidal categories.

Definition 6.2.8. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be an abelian monoidal category.

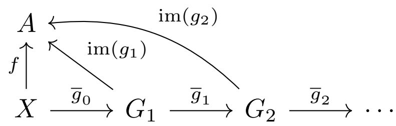
1. Let (A, μ, η) be a unital algebra in $\mathcal{C}$, and let (X, f) be a subobject of A.

   a) We define $g_0 := f, G_0 := X$, and inductively

   $$g_n := \mu \circ (\mathrm{id}_A \otimes \mathrm{im}(g_{n-1})) : A \otimes G_{n-1} \to A \text{ and } G_n := \mathrm{Im}(g_n). \tag{6.75}$$

   Setting $\overline{g}_n := \mathrm{coim}(g_{n+1}) \circ (\eta \otimes \mathrm{id}_{G_n}) \circ \lambda_{G_n}^{-1}$, we see that $\mathrm{im}(g_{n+1}) \circ \overline{g}_n = \mathrm{im}(g_n)$ by using the fact that η is a unit for (A, μ) (6.4), and the fact that the left unital is a natural isomorphism (3.2).

   We thus obtain a commutative diagram

   (6.76)

   Define a category $\mathcal{I}_{\text{left}}$ that has an object $\star_n$ for every $n \geq 0$, and for which there is a unique morphism $i_n : \star_n \to \star_{n+1}$ (and all other morphisms are either identities or compositions of these morphisms). It is easy to see that this category is filtered.

We define $G_{\text{left}} : \mathcal{I}_{\text{left}} \to \mathcal{C} : \star_n \mapsto G_n$ and $i_n \mapsto \overline{g}_n$, and we denote the colimit of this functor (in $\mathcal{C}^{\text{ind}}$ if it does not exist in $\mathcal{C}$) $\text{colim}(G_{\text{left}}) = \text{colim}_n(G_n)$ and the corresponding morphisms $\underline{g}_n : G_n \to \text{colim}(G_{\text{left}})$.

The above diagram shows that $(A, \text{im}(g_n))$ is a cocone on G_{left}. We thus obtain a unique morphism $i_{\text{left}} : \text{colim}(G_{\text{left}}) \to A$ such that $i \circ \underline{g}_n = \text{im}(g_n)$ for all $n \geq 0$.

b) Similarly, we define a functor $G_{\text{right}} : \mathcal{I}_{\text{right}} \to \mathcal{C}$ where $\mathcal{I}_{\text{right}}$ is a filtered category obtained by replacing g_n by $\mu \circ (\text{im}(g_{n-1}) \otimes \text{id}_A)$ in the above. The induced morphism is denoted $i_{\text{right}} : \text{colim}(G_{\text{right}}) \to A$.

c) We define $g_0 := f, G_0 := X$ as before, and we set

$$g_{2n} := \mu \circ (\text{im}(g_{2n-1}) \otimes \text{id}_A), g_{2n-1} := \mu \circ (\text{id}_A \otimes \text{im}(g_{2n-2})), \text{ and } G_n := \text{Im}(g_n). \tag{6.77}$$

We can once again define morphisms $\overline{g}_n : G_n \to G_{n+1}$ such that $\text{im}(g_{n+1}) \circ \overline{g}_n = \text{im}(g_n)$ by setting $\overline{g}_{2n} = \text{coim}(g_{2n+1}) \circ (\eta \otimes \text{id}_{G_{2n-1}}) \circ \lambda^{-1}_{G_{2n-1}}$ and $\overline{g}_{2n-1} = \text{coim}(g_{2n}) \circ (\text{id}_{G_{2n-2}} \otimes \eta) \circ \rho^{-1}_{G_{2n-2}}$, a category $\mathcal{I}$ modelling this sequence of monomorphisms, a functor $G : \mathcal{I} \to \mathcal{C}$ mapping the category to the sequence, and a pair $(\text{colim}(G), i)$ of the filtered colimit of G together with a uniquely induced morphism $i : \text{colim}(G) \to A$.

2. Let (A, μ, η) be a counital coalgebra in $\mathcal{C}$, and let (X, f) be a quotient of A.

a) We define $h_0 := f, H_0 := X$, and inductively

$$h_n = (\text{id}_A \otimes \text{coim}(h_{n-1})) \circ \mu : A \to A \otimes H_{n-1} \circ \mu \text{ and } H_n := \text{Coim}(h_n). \tag{6.78}$$

Set $\overline{h}_n := \lambda_{H_n} \circ (\eta \otimes \text{id}_{H_n}) \circ \text{im}(h_{n+1})$, we see that $\overline{h}_n \circ \text{coim}(h_{n+1}) = \text{coim}(h_n)$.

We thus obtain a commutative diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\overline{h}_2} & H_2 & \xrightarrow{\overline{h}_1} & H_1 & \xrightarrow{\overline{h}_0} & X \\ & & \nwarrow & & \nwarrow & & \uparrow f \\ & & \text{coim}(h_2) & & \text{coim}(h_1) & & A \end{array} \tag{6.79}$$

As before we define a functor $H_{\text{left}} : \mathcal{J}_{\text{left}} \to \mathcal{C}$ where $\mathcal{J}_{\text{left}}$ is a cofiltered category consisting of objects modelling the horizontal part of the above diagram, and where the functor maps this category to the horizontal sequence.

We denote the limit of this functor $\lim(H_{\text{left}})$, and the induced morphism $q_{\text{left}} : A \to \lim(H_{\text{left}})$.

b) We can define a functor $H_{\text{right}} : \mathcal{J}_{\text{right}} \to \mathcal{C}$. We obtain the limit $\lim(H_{\text{right}})$ and the induced morphism $q_{\text{right}} : A \to \lim(H_{\text{right}})$.

c) We can define a functor $H : \mathcal{J} \to \mathcal{C}$. We obtain the limit $\lim(H)$ and the induced morphism $q : A \to \lim(H)$.

We will prove that these filtered colimits are the ideals generated by the corresponding subobjects. One technical result we need for this, is that the induced morphism $\text{colim}_n(G_n) \to A$ is a monomorphism.

Lemma 6.2.9. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a small abelian monoidal category. The morphisms $i_{\text{left}}, i_{\text{right}}, i$ defined above are monomorphisms, and the morphisms $q_{\text{left}}, q_{\text{right}}, q$ are epimorphisms.*

Proof omitted. This follows from the fact that the ind-cocompletion of a (small) abelian category is a Grothendieck category (see for example this nLab page on Grothendieck categories, [aut25b]), which implies that the short exact sequences

$$0 \longrightarrow G_n \xrightarrow{\text{im}(g_n)} A \xrightarrow{\text{coker}(g_n)} \text{Coker}(g_n) \longrightarrow 0 \tag{6.80}$$

get mapped to the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{colim}_n(G_n) & \xrightarrow{\mathrm{colim}_n(\mathrm{im}(g_n))} & \mathrm{colim}(A) & \xrightarrow{\mathrm{colim}_n(\mathrm{coker}(g_n))} & \mathrm{colim}_n(\mathrm{Coker}(g_n)) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathrm{colim}(G_{\mathrm{left}}) & \xrightarrow{i_{\mathrm{left}}} & A & \xrightarrow{\mathrm{colim}_n(\mathrm{coker}(g_n))} & \mathrm{colim}_n(\mathrm{Coker}(g_n)) & \longrightarrow & 0 \end{array} . \tag{6.81}$$

We can then conclude that i_{left} is a monomorphism. ∎

Proposition 6.2.10. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms.*

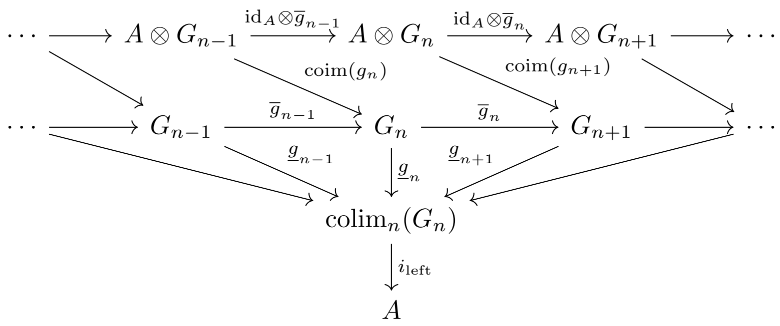
1. *Let (A, μ, η) be a unital algebra in $\mathcal{C}$, and let (X, f) be a subobject of A.*
   a) *$(\mathrm{colim}(G_{\mathrm{left}}), i_{\mathrm{left}})$ is a left ideal, $(\mathrm{colim}(G_{\mathrm{right}}), i_{\mathrm{right}})$ is a right ideal, and $(\mathrm{colim}(G), i)$ is a double ideal in (A, μ, η).*
   b) *$(\mathrm{colim}(G_{\mathrm{left}}), i_{\mathrm{left}})$ is the left ideal generated by (X, f), $(\mathrm{colim}(G_{\mathrm{right}}), i_{\mathrm{right}})$ is the right ideal generated by (X, f), and $(\mathrm{colim}(G), i)$ is the double ideal generated by (X, f).*
   c) *If (A, μ) is associative, then $\mathrm{colim}(G_{\mathrm{left}}) = \mathrm{Im}(\mu \circ (\mathrm{id}_A \otimes f)), i_{\mathrm{left}} = \mathrm{im}(\mu \circ (\mathrm{id}_A \otimes f))$, and $\mathrm{colim}(G_{\mathrm{right}}) = \mathrm{Im}(\mu \circ (f \otimes \mathrm{id}_A)), i_{\mathrm{right}} = \mathrm{im}(\mu \circ (f \otimes \mathrm{id}_A))$.*
   d) *If, in addition, the category is braided, and (A, μ) is commutative or anti-commutative, then $\mathrm{colim}(G) = \mathrm{colim}(G_{\mathrm{left}}) = \mathrm{colim}(G_{\mathrm{right}})$, and $i = i_{\mathrm{left}} = i_{\mathrm{right}}$.*
2. *Let (A, μ, η) be a counital coalgebra in $\mathcal{C}$, and let (X, f) be a quotient of A.*
   a) *$(\lim(H_{\mathrm{left}}), q_{\mathrm{left}})$ is a left coideal, $(\lim(H_{\mathrm{right}}, q_{\mathrm{right}}))$ is a right coideal, and $(\lim(H), q)$ is a double coideal in (A, μ, η).*
   b) *$(\lim(H_{\mathrm{left}}), q_{\mathrm{left}})$ is the left coideal generated by (X, f), $(\lim(H_{\mathrm{right}}, q_{\mathrm{right}}))$ is the right coideal generated by (X, f), and $(\lim(H), q)$ is the double coideal generated by (X, f).*
   c) *If (A, μ) is coassociative, then $\lim(H_{\mathrm{left}}) = \mathrm{Coim}((\mathrm{id}_A \otimes f) \circ \mu), q_{\mathrm{left}} = \mathrm{coim}((\mathrm{id}_A \otimes f) \circ \mu)$, and $\lim(H_{\mathrm{right}}) = \mathrm{Coim}((f \otimes \mathrm{id}_A) \circ \mu), q_{\mathrm{right}} = \mathrm{coim}((f \otimes \mathrm{id}_A) \circ \mu)$.*
   d) *If, in addition, the category is braided, and (A, μ) is cocommutative or anti-cocommutative, then $\lim(H) = \lim(H_{\mathrm{left}}) = \lim(H_{\mathrm{right}}) = \mathrm{Coim}((\mathrm{id}_A \otimes f) \circ \mu)$, and $q = q_{\mathrm{left}} = q_{\mathrm{right}} = \mathrm{coim}((\mathrm{id}_A \otimes f) \circ \mu)$.*

Proof. We will first prove (1a).

Lemma 6.2.9 shows that i_{left} is a monomorphism. All that we have to check is that (6.35) holds. As $\mathcal{C}$ is rigid, we know that $A \otimes -$ is a left adjoint, which implies that it preserves colimits through Theorem 1.4.3. We thus find $\mathrm{colim}_n(A \otimes G_n) = A \otimes \mathrm{colim}_n(G_n)$, where the colimit is taken with regard to the filtration

$$\cdots \longrightarrow A \otimes G_{n-1} \xrightarrow{\mathrm{id}_A \otimes \overline{g}_{n-1}} A \otimes G_n \xrightarrow{\mathrm{id}_A \otimes \overline{g}_n} A \otimes G_{n+1} \longrightarrow \cdots \tag{6.82}$$

We then find a commutative diagram



, (6.83)

where the squares commute due to

$$\begin{aligned} \mathrm{im}(g_{n+1}) \circ \overline{g}_n \circ \mathrm{coim}(g_n) &= \mathrm{im}(g_n) \circ \mathrm{coim}(g_n) = g_n \\ &= \mu \circ (\mathrm{id}_A \otimes \mathrm{im}(g_{n-1})) \\ &= \mu \circ (\mathrm{id}_A \otimes (\mathrm{im}(g_n) \circ \overline{g}_{n-1})) \\ &= g_{n+1} \circ (\mathrm{id}_A \otimes \overline{g}_{n-1}) \\ &= \mathrm{im}(g_{n+1}) \circ \mathrm{coim}(g_{n+1}) \circ (\mathrm{id}_A \otimes \overline{g}_{n-1}) \end{aligned} \quad . \tag{6.84}$$

This implies that we have cocones $(\mathrm{colim}_n(G_n), \underline{g}_{n+1} \circ \mathrm{coim}(g_{n+1}))$ and (A, g_{n+1}), and hence uniquely induced morphisms

$$\mu_{\mathrm{colim}_n(G_n)} : A \otimes \mathrm{colim}_n(G_n) \to \mathrm{colim}_n(G_n) \text{ and } \zeta : A \otimes \mathrm{colim}_n(G_n) \to A \tag{6.85}$$

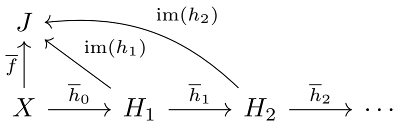
such that $\underline{g}_{n+1} \circ \mathrm{coim}(g_{n+1}) = \mu_{\mathrm{colim}_n(G_n)} \circ (\mathrm{id}_A \otimes \underline{g}_n)$ and $g_{n+1} = \zeta \circ (\mathrm{id}_A \otimes \underline{g}_n)$ for all n.

It is then clear that $\zeta = \mu \circ (\mathrm{id}_A \otimes i_{\mathrm{left}})$ as $\mu \circ (\mathrm{id}_A \otimes i_{\mathrm{left}}) \circ (\mathrm{id}_A \otimes \underline{g}_n) = \mu \circ (\mathrm{id}_A \otimes \mathrm{im}(g_n)) = g_{n+1}$ for all n. However, we also find $\zeta = i_{\mathrm{left}} \circ \mu_{\mathrm{colim}_n(G_n)}$ as $i_{\mathrm{left}} \circ \mu_{\mathrm{colim}_n(G_n)} \circ (\mathrm{id}_A \otimes \underline{g}_n) = i_{\mathrm{left}} \circ \underline{g}_{n+1} \circ \mathrm{coim}(g_{n+1}) = g_{n+1}$.

As ζ is unique, we conclude that (6.35) holds.

The proof for the right ideal and double ideal are identical, where we use that $\mathrm{colim}_n(G_n) = \mathrm{colim}_{2n}(G_{2n}) = \mathrm{colim}_{2n+1}(G_{2n+1})$ for the double ideal.

We will now prove (1b). First, note that it is trivial that $(\mathrm{colim}(G_{\mathrm{left}}), i_{\mathrm{left}})$ contains (X, f) because $G_0 = X$ and $g_0 = f$. Let (J, j) be any other left ideal that contains (X, f). Replacing (A, μ, η) by (J, μ_J, η) and f by the induced morphism $\overline{f} : X \to J$ such that $f = j \circ \overline{f}$, we find a commutative diagram



(6.86)

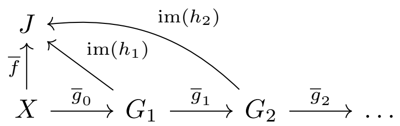
We have

$$j \circ h_1 = j \circ \mu_J \circ (\mathrm{id}_A \otimes \overline{f}) = \mu \circ (\mathrm{id}_A \otimes j) \circ (\mathrm{id}_A \otimes \overline{f}) = \mu \circ (\mathrm{id}_A \otimes f) = g_1, \tag{6.87}$$

and inductively

$$j \circ h_n = j \circ \mu_J \circ (\mathrm{id}_A \otimes \mathrm{im}(h_{n-1})) = \mu \circ (\mathrm{id}_A \otimes j) \circ (\mathrm{id}_A \otimes \mathrm{im}(h_{n-1})) = \mu \circ (\mathrm{id}_A \otimes \mathrm{im}(g_{n-1})) = g_n. \tag{6.88}$$

As a consequence, we find $H_n = \mathrm{Im}(h_n) = \mathrm{Im}(g_n) = G_n$ and $j \circ \mathrm{im}(h_n) = \mathrm{im}(g_n)$. This implies that we have a cocone



, (6.89)

and a uniquely induced morphism $\overline{i}_{\mathrm{left}} : \mathrm{colim}_n(G_n) \to J$ such that $j \circ \overline{i}_{\mathrm{left}} = i_{\mathrm{left}}$.

The proofs for the right and double ideal are again identical.

For (1c), we have

$$\begin{aligned} g_2 \circ (\mathrm{id}_A \otimes \mathrm{coim}(g_1)) &= \mu \circ (\mathrm{id}_A \otimes g_1) \\ &= \mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \otimes f)) \\ &= \mu \circ (\mu \otimes \mathrm{id}_A) \circ \alpha^{-1}_{(A,A,A)} \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \otimes f)) \\ &= \mu \circ (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A \otimes A} \otimes f) \circ \alpha^{-1}_{(A,A,X)} \\ &= \mu \circ (\mathrm{id}_A \otimes f) \circ (\mu \otimes \mathrm{id}_X) \circ \alpha^{-1}_{(A,A,X)} \\ &= g_1 \circ (\mu \otimes \mathrm{id}_X) \circ \alpha^{-1}_{(A,A,X)} \end{aligned} \quad . \tag{6.90}$$

This implies that $g_2 \circ \mathrm{epi} = g_1 \circ \mathrm{epi}$ (using that $\mathrm{id} \otimes \mathrm{epi}$ and $\mathrm{epi} \otimes \mathrm{id}$ are epimorphisms because the category is rigid, and that μ is a split epimorphism because the algebra is unital). In particular, $\mathrm{im}(g_2) = \mathrm{im}(g_1)$ and hence $\mathrm{im}(g_n) = \mathrm{im}(g_1)$ and $\mathrm{Im}(g_n) = \mathrm{Im}(g_1)$ for all $n \geq 1$. The sequence of monomorphisms then becomes

$$X \longrightarrow G_1 \xrightarrow{\mathrm{id}_{G_1}} G_1 \xrightarrow{\mathrm{id}_{G_1}} \dots \tag{6.91}$$

The colimit of this diagram is G_1.

Finally, we will prove (1d). Suppose that $\mathcal{C}$ is equipped with a braiding γ, and that (A, μ) is commutative. We then find

$$\mu \circ (\mathrm{id}_A \otimes f) = \mu \circ \gamma_{(A,A)} \circ (\mathrm{id}_A \otimes f) = \mu \circ (f \otimes \mathrm{id}_A) \circ \gamma_{(A,X)}, \tag{6.92}$$

and similarly

$$g_n = \mu \circ (\mathrm{id}_A \otimes \mathrm{im}(g_{n-1})) = \mu \circ (\mathrm{im}(g_{n-1}) \otimes \mathrm{id}_A) \circ \gamma_{(A,G_{n-1})}. \tag{6.93}$$

This implies that the sequences of monomorphisms, and hence colimits, are equal. ■

Remark 6.2.11. In Remark 1.3.14, we already showed that arbitrary intersections of collections of subobjects exist and can be expressed as filtered colimits. It would therefore have been possible to define the ideal generated by a subobject as the intersection of all ideals containing it. Consequently, the result above is of interest only because it provides a construction of the ideal generated by a subobject using a *countable* filtered colimit, and because the objects appearing in this colimit are much better understood than arbitrary ideals.

We have not yet discussed ideals generated by objects in the context of non-unital algebras. We could try to find a technical construction generalising Definition 6.2.8, but we could also use the theory of unital hulls we have already introduced. More specifically, we will now show that Lemma 6.2.6 and Proposition 6.2.7 allow us to extend the above result to the setting of non-unital algebras. The idea to use unital hulls for this purpose, which was the author's sole motivation for developing any of the preceding discussion on unital hulls, was suggested by Prof. Dr. Tom De Medts.

Corollary 6.2.12. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a pre-abelian monoidal category such that the monoidal product is bilinear on morphisms.*

1. *Let (A, μ) be an algebra in $\mathcal{C}$, and let (A_u, μ_u, η_u) be its unital hull.*

   Let (X_u, f_u) be a subobject of A_u, and let (I_u, i_u) be the left, right, or double ideal generated by (X_u, f_u) in (A_u, μ_u). The ideal $(I, i) = \overline{\mathrm{UnitHull}}_A(I_u, i_u)$ in (A, μ) is the left, right, or double ideal generated by the subobject $(X, f) = \overline{\mathrm{UnitHull}}_A(X_u, f_u)$ of A.

   This implies that, when the category is additionally rigid, ideals generated by subobjects can be obtained through the filtered colimit construction given in Definition 6.2.8.

2. *Let (A, μ) be a coalgebra in $\mathcal{C}$, and let (A_u, μ_u, η_u) be its unital hull.*

   Let (X_u, f_u) be a quotient of A_u, and let (Q_u, q_u) be the left, right, or double coideal generated by (X_u, f_u) over (A_u, μ_u). The coideal $(Q, q) = \overline{\mathrm{CounitHull}}_A(Q_u, q_u)$ over (A, μ) is the left, right, or double coideal generated by the quotient $(X, f) = \overline{\mathrm{CounitHull}}(X_u, f_u)$ of A.

   This implies that, when the category is additionally rigid, coideals generated by quotients can be obtained through the cofiltered limit construction given in Definition 6.2.8.

Proof. The ideal generated by (X, f) in (A, μ) is an initial object in the full subcategory of $\mathrm{Sub}(A)$ consisting of ideals which contain (X, f). $\overline{\mathrm{UnitHull}}_A \circ \mathrm{UnitHull}_A = \mathrm{id}_{\mathrm{Sub}(A)}$ (Lemma 6.2.6) shows that $\overline{\mathrm{UnitHull}}_A$ preserves such initial objects, as this shows that any ideal in (A, μ) containing (X, f) can be obtained by applying $\overline{\mathrm{UnitHull}}_A$ to an ideal in (A_u, μ_u) containing (X_u, f_u), ■

Our discussion of ideals generated by subobjects allows us to define finitely generated algebras in our general setting. A more restrictive version of this definition for unital associative commutative algebras can be found in [Ven23].

Definition 6.2.13 (Finitely generated algebras). Let $\mathcal{C}$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms.

1. An algebra (A, μ) in $\mathcal{C}^{\text{ind}}$ is called *finitely generated* if there is a subobject (X, f) of A such that $X \in \text{Ob}(\mathcal{C})$ (which can be interpreted as X being "finite-dimensional"), and such that the (double) ideal generated by (X, f) is isomorphic to (A, μ).
2. A coalgebra (A, μ) in $\mathcal{C}^{\text{ind}}$ is called finitely generated if there is a quotient (X, f) of A such that $X \in \text{Ob}(\mathcal{C})$, and such that the (double) coideal generated by (X, f) is isomorphic to (A, μ).

The tensor algebra

The above discussion also allows us to define the tensor, symmetric, and exterior algebra on any object.

Example 46 (Tensor, symmetric, and exterior algebra, [EGNO15, Remark 9.9.6] and [Ven23, § 2.3]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be an abelian rigid monoidal category such that the monoidal product is bilinear on morphisms, and let $A \in \text{Ob}(\mathcal{C})$ be any object. We define $T_0(A) = T_0 A := \mathbb{1}$, and inductively

$$T_n(A) = T_n A := T_{n-1}(A) \oplus A^{\otimes n}. \tag{6.94}$$

We obtain the filtered diagram

$$\mathbb{1} = T_0(A) \xrightarrow{\text{inc}_{T_0(A)}} A = T_1(A) \xrightarrow{\text{inc}_{T_1(A)}} T_2(A) \xrightarrow{\text{inc}_{T_2(A)}} \dots \tag{6.95}$$

The colimit of this diagram in $\mathcal{C}^{\text{ind}}$ is denoted

$$T(A) = TA = \bigoplus_{n=0}^{\infty} A^{\otimes n}, \tag{6.96}$$

and the morphisms into the colimit are denoted $i_n : T_n(A) \to T(A)$. As $\mathcal{C}$ is rigid, we know that the monoidal product preserves colimits, hence that $T(A) \otimes T(A)$ is the colimit of $T_n(A) \otimes T_m(A)$ with morphisms $i_{nm} : T_n(A) \otimes T_m(A) \to T(A) \otimes T(A)$. However, we also have morphisms

$$\mu_{nm} : T_n(A) \otimes T_m(A) \to T(A) \text{ obtained by mapping } A^{\otimes n} \otimes A^{\otimes m} \text{ to } A^{\otimes(n+m)}. \tag{6.97}$$

We find a cocone $(T(A), \mu_{nm})$ on $T_n(A) \otimes T_m(A)$, hence a uniquely induced morphism

$$\mu : T(A) \otimes T(A) \to T(A) \text{ such that } \mu \circ i_{nm} = \mu_{nm}. \tag{6.98}$$

The algebra $(T(A), \mu)$ is called the *tensor algebra* on A. This algebra is associative and unital with unit i_0.

Suppose that, in addition, $\mathcal{C}$ is equipped with a braiding γ.

We define the *symmetric algebra* $S(A) = SA$ in $\mathcal{C}^{\text{ind}}$ as the quotient algebra of the tensor algebra $(T(A), \mu)$ over the ideal generated by $\text{Im}(\text{id}_{A \otimes A} - \gamma_{(A,A)})$.

We define the *exterior algebra* $\wedge(A) = \wedge A$ as the quotient of $T(A)$ over the ideal generated by $\text{Im}(\text{id}_{A \otimes A} + \gamma_{A \otimes A})$.

Remark 6.2.14. Our definitions of the tensor, symmetric, and exterior algebras differ slightly from those commonly found in the literature (see, for example, [EGNO15; Ven23]).

Let a_n be the morphism $T_n(A) \to T(A)$ obtained by extending the morphism $\text{id}_{A \otimes A} - \gamma_{(A,A)} : A \otimes A \to A \otimes A$ (applying the, unnormalised, skew-symmetriser in every degree). This gives us a cocone on $(T_n(A))_{n \in \mathbb{N}}$, hence a uniquely defined morphism $a : T(A) \to T(A)$ such that $a \circ i_n = a_n$ for all n. $(\text{Ker}(a), \ker(a))$ is an ideal in $(T(A), \mu)$ as $a \circ \mu \circ (\ker(a) \otimes \ker(a)) = 0$. The exterior algebra can then be defined as the quotient of $(T(A), \mu)$ over $(\text{Ker}(a), \ker(a))$. This definition coincides with the above one if the characteristic of the field is not two.

We can also define the symmetric algebra and the exterior algebra as filtered colimits of $S_n(A) = \oplus_{k=0}^{n} S^k(A)$ and $\wedge_n(A) = \oplus_{k=0}^{n} \wedge^k (A)$.

6.3 Lie algebras in categories

Next, we consider Lie algebras in the context of monoidal categories, as discussed in [EGNO15; Eti18; Ven23; Kan24]. This requires a categorification of the classical Lie algebra axioms. Fortunately, the axioms for Lie algebras (over fields of characteristic not equal to two) are linear, which allows for a straightforward generalisation to the (braided) monoidal setting.

Definition 6.3.1 ((Operadic) Lie algebras, [EGNO15, Exercise 9.9.7] and [Eti18, Definition 4.1]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive symmetric monoidal category such that the monoidal product is bilinear on morphisms. An algebra (A, μ) in $\mathcal{C}$ is called an (*operadic*) *Lie algebra* if it is anti-commutative, and the following *Jacobi identity* holds

$$\mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2) = 0, \tag{6.99}$$

where $\sigma : A \otimes A \otimes A \to A \otimes A \otimes A$ is the automorphism corresponding to $(123) \in S_3$, i.e. $\sigma = \gamma_{(A,A\otimes A)}$.

Remark 6.3.2. We will refer to operadic Lie algebras simply as Lie algebras. However, some authors reserve this term for operadic Lie algebras that satisfy the Poincaré-Birkhoff-Witt (PBW) theorem. It is known (see [Eti18]) that in most symmetric tensor categories, not every operadic Lie algebra satisfies the PBW theorem, unlike in the classical case.

Example 47. Lie algebras in the category of vector spaces over a field $\mathbb{K}$ coincide with the classical notion of Lie algebras over $\mathbb{K}$.

Example 48 (Commutator brackets, [EGNO15, Exercise 9.9.7 (i)]). Let $\mathcal{C}$ be a pre-additive symmetric monoidal category, and let (A, μ) be an associative algebra. We define the *commutator bracket* as

$$\mu_L := \mu \circ (\mathrm{id}_{A\otimes A} - \gamma_{(A,A)}). \tag{6.100}$$

We claim that (A, μ_L) is a Lie algebra. It is immediately clear that (A, μ_L) is anti-commutative. Let us now check that the Jacobi identity (6.99) holds.

Let us first note that

$$(\gamma_{(A,A)} \otimes \mathrm{id}_A) \circ \gamma_{(A,A\otimes A)} = \gamma_{(A,A\otimes A)} \circ (\mathrm{id}_A \otimes \gamma_{(A,A)}) \tag{6.101}$$

through the Yang-Baxter theorem 3.6.6.

This implies that

$$\begin{aligned} &\mu_L \circ (\mathrm{id}_A \otimes \mu_L) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2) \\ &= \Big(\mu \circ (\mathrm{id}_A \otimes \mu) - \mu \circ (\mu \otimes \mathrm{id}_A) \circ \gamma_{(A,A\otimes A)}\Big) \circ (\mathrm{id}_{A\otimes A\otimes A} - \mathrm{id}_A \otimes \gamma_{(A,A)}) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2) \end{aligned}. \tag{6.102}$$

As (A, μ) is associative, we then find

$$\begin{aligned} &\mu_L \circ (\mathrm{id}_A \otimes \mu_L) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2) \\ &= \mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_{A\otimes A\otimes A} - \gamma_{(A,A\otimes A)}) \circ (\mathrm{id}_{A\otimes A\otimes A} - \mathrm{id}_A \otimes \gamma_{(A,A)}) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2) \\ &= \mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_{A\otimes A\otimes A} - \gamma_{(A,A)} \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A\otimes A\otimes A} - \sigma) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2) \\ &= 0 \end{aligned}. \tag{6.103}$$

In particular, for every object $X \in \mathrm{Ob}(\mathcal{C})$, there is a Lie algebra structure on the internal endomorphism algebra $\underline{\mathrm{End}}(X)$, which we denote $(\mathfrak{gl}(X), \nabla_L)$.

Example 49 (Orthogonal and symplectic Lie algebras, [EGNO15, Exercise 9.9.7 (ii)]). Let $\mathcal{C}$ be a symmetric tensor category and let $A \in \mathrm{Ob}(\mathcal{C})$ be equipped with an isomorphism $\beta : A \to A^*$ such that $\beta^* = \beta$ or $\beta^* = -\beta$ (more specifically $\beta^* \circ \omega_A = \pm\beta$, where $\omega : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}{}^{**}$ is the natural isomorphism defined in Corollary 3.6.8). We then define an algebra $(\wedge^2 A, \mu)$ through

$$\begin{array}{ccccccc} \wedge^2 A \otimes \wedge^2 A & & \overset{\mu}{\dashrightarrow} & & & & \wedge^2 A \\ \downarrow{\scriptstyle \mathrm{im}(a_2)\otimes\mathrm{im}(a_2)} & & & & & & \uparrow{\scriptstyle \mathrm{coim}(a_2)} \\ (A \otimes A) \otimes (A \otimes A) & \xrightarrow{(\mathrm{id}_A \otimes \beta)\otimes(\mathrm{id}_A \otimes \beta)} & (A \otimes A^*) \otimes (A \otimes A^*) & \xrightarrow{\nabla_L} & A \otimes A^* & \xrightarrow{\mathrm{id}_A \otimes \beta^{-1}} & A \otimes A \end{array} . \tag{6.104}$$

These algebras are Lie algebras (but we will not work this out here, as this is somewhat tedious and not particularly insightful).

It is called the *orthogonal Lie algebra* $\mathfrak{o}(A, \beta)$ if $\beta^* = \beta$, and the *symplectic Lie algebra* $\mathfrak{sp}(A, \beta)$ if $\beta^* = -\beta$.

Example 50 (Lie superalgebras). Let $\mathbb{K}$ be some field. Lie algebras in $\mathbf{sVect}_{\mathbb{K}}$ are called *Lie superalgebras* in the literature (see, for example, [EGNO15; Kan24]). If $V \in \mathrm{Ob}(\mathbf{sVect}_{\mathbb{K}})$ has even dimension n_e and odd dimension n_o, then we also write $\mathfrak{gl}(V) = \mathfrak{gl}(n_e|n_o)$, $\mathfrak{o}(V, \beta) = \mathfrak{osp}(n_e|n_o, \beta)$, $\mathfrak{sp}(V, \beta) = \mathfrak{osp}(n_o|n_e, \beta)$ (see, for example, [EGNO15; Kan24]).

6.4 Hopf algebras

In Section 4.5.3, we discussed the classification of pre-Tannakian symmetric tensor categories of moderate growth. In particular, this classification highlights the importance of Hopf algebras and their categories of modules or comodules in a relatively small collection of exotic symmetric tensor categories. In this section, we will study bialgebras and Hopf algebras, and we will devote considerable attention to proving that the category of modules over a Hopf algebra forms a symmetric tensor category.

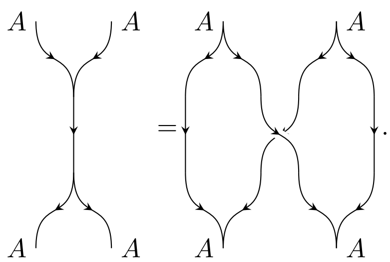
6.4.1 Bialgebras and Hopf algebras

Before introducing Hopf algebras, we first introduce bialgebras, which are objects equipped with both an algebra and a coalgebra structure, such that these structures are compatible.

Definition 6.4.1 (Bialgebras, [EGNO15, Definition 5.2.2 and Exercise 9.9.7 (vi)]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive braided monoidal category such that the monoidal product is bilinear on morphisms. A *bialgebra* in $\mathcal{C}$ is a tuple (A, ∇, Δ) such that $\nabla : A \otimes A \to A$, $\Delta : A \to A \otimes A$, called the *multiplication* and *comultiplication* respectively, are morphisms making the following diagram commute

$$\begin{array}{ccc} A \otimes A \xrightarrow{\ \nabla\ } A \xrightarrow{\ \Delta\ } & & A \otimes A \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & \uparrow{\scriptstyle \nabla\otimes\nabla} \\ (A \otimes A) \otimes (A \otimes A) & \xrightarrow{\mathrm{id}_A \otimes \gamma_{(A,A)} \otimes \mathrm{id}_A} & (A \otimes A) \otimes (A \otimes A) \end{array} . \tag{6.105}$$

Graphically, this is



(6.106)

Note that this just states that (A, ∇) is an algebra, (A, Δ) is a coalgebra, and that Δ is an algebra morphism $(A, \nabla) \to (A, \nabla) \otimes (A, \nabla)$ (or equivalently that ∇ is a coalgebra morphism $(A, \Delta) \otimes (A, \Delta) \to (A, \Delta)$).

A bialgebra (A, ∇, Δ) is called *associative* if (A, ∇) is an associative algebra and (A, Δ) is a coassociative coalgebra, i.e. if the following diagrams commute

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \xrightarrow{\nabla \otimes \mathrm{id}_A} & A \otimes A & & \\ \downarrow{\scriptstyle \alpha_{(A,A,A)}} & & & \searrow{\scriptstyle \nabla} & \\ & & & & A \\ & & & \nearrow{\scriptstyle \nabla} & \\ A \otimes (A \otimes A) & \xrightarrow{\mathrm{id}_A \otimes \nabla} & A \otimes A & & \end{array} \qquad \begin{array}{ccccc} & & A \otimes A & \xrightarrow{\Delta \otimes \mathrm{id}_A} & (A \otimes A) \otimes A \\ & \nearrow{\scriptstyle \Delta} & & & \\ A & & & & \downarrow{\scriptstyle \alpha_{(A,A,A)}} \\ & \searrow{\scriptstyle \Delta} & & & \\ & & A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \Delta} & A \otimes (A \otimes A) \end{array} . \tag{6.107}$$

A bialgebra (A, ∇, Δ) is called *unital* if there exist morphisms $\eta : \mathbb{1} \to A, \varepsilon : A \to \mathbb{1}$ such that (A, ∇, η) and (A, Δ, ε) are unital and counital respectively, making the following diagrams commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\nabla} & A \\ {\scriptstyle \varepsilon \otimes \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\lambda_{\mathbb{1}}} & \mathbb{1} \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\lambda_{\mathbb{1}}^{-1}} & \mathbb{1} \otimes \mathbb{1} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta \otimes \eta} \\ A & \xrightarrow{\Delta} & A \otimes A \end{array} . \tag{6.108}$$

This states that ∇ is in fact a unital coalgebra morphism, and that Δ is in fact a unital algebra morphism.

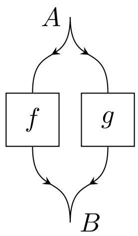
A bialgebra is called *commutative* if the underlying algebra (A, ∇) is commutative, and *cocommutative* if the underlying coalgebra (A, Δ) is cocommutative.

Bialgebras allow us to introduce a new multiplication on morphisms, the convolution.

Definition 6.4.2 (Convolution). Let (A, ∇_A, Δ_A) and (B, ∇_B, Δ_B) be two bialgebras in a pre-additive braided monoidal category with a bilinear monoidal product. We define the *convolution* of two morphisms $f, g : A \to B$ as

$$f \star g := \nabla_B \circ (f \otimes g) \circ \Delta_A : A \to B. \tag{6.109}$$

Graphically, this is

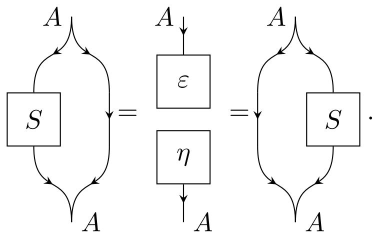
$f \star g =$  . (6.110)

It is clear that this operation has the unit $\eta_B \circ \varepsilon_A$ when $(A, \Delta_A, \varepsilon_A)$ is counital and (B, ∇_B, η_B) is unital, and it is also clear that this operation is associative when (A, Δ_A) is coassociative and (B, ∇_B) is associative.

Definition 6.4.3 (Hopf algebras, [EGNO15, Definition 5.3.10 and Exercise 9.9.7 (vi)]). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive braided monoidal category such that the monoidal product is bilinear on morphisms. A *Hopf algebra* is a tuple $(A, \nabla, \eta, \Delta, \varepsilon, S)$ such that $(A, \nabla, \eta, \Delta, \varepsilon)$ is a unital associative bialgebra, and such that $S : A \to A$, called the *antipode*, makes the following diagram commute

$$\begin{array}{ccccccc} & & A \otimes A & \xrightarrow{S \otimes \mathrm{id}_A} & A \otimes A & & \\ & \nearrow{\scriptstyle \Delta} & & & & \searrow{\scriptstyle \nabla} & \\ A & & \xrightarrow{\varepsilon} & \mathbb{1} & \xrightarrow{\eta} & & A \\ & \searrow{\scriptstyle \Delta} & & & & \nearrow{\scriptstyle \nabla} & \\ & & A \otimes A & \xrightarrow{\mathrm{id}_A \otimes S} & A \otimes A & & \end{array} . \tag{6.111}$$

Graphically, this is



(6.112)

Remark 6.4.4. Note that the definitions of bialgebras and Hopf algebras are self-dual, which implies that in categories with duals, the dual of a bialgebra is a bialgebra, and the dual of a Hopf algebra is a Hopf algebra.

Example 51 (Group algebras). Let G be a group and let $\mathbb{K}$ be some field. The group algebra $\mathbb{K}G$ can be equipped with a $\mathbf{Vect}_{\mathbb{K}}$-Hopf algebra structure

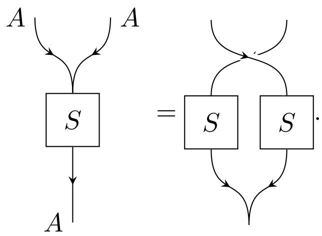
1. as a vector space $\mathbb{K}G$ is the free $\mathbb{K}$-vector space on G,
2. the multiplication and the unit on $\mathbb{K}G$ are induced by the multiplication and unit on G,
3. the comultiplication is given on the basis of $\mathbb{K}G$ by $g \mapsto g \otimes g$, i.e. $\sum_{g\in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g g \otimes g$,
4. the counit on $\mathbb{K}G$ is given by $\varepsilon : \mathbb{K}G \to \mathbb{K} : \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g$,
5. the antipode on $\mathbb{K}G$ is given by $S : \mathbb{K}G \to \mathbb{K}G : \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g g^{-1}$.

It is then easy to check that (6.105), (6.107), (6.108), and (6.111) hold.

Example 52 (Coordinate algebras of affine algebraic groups). The coordinate algebras of affine algebraic groups (in the classical setting, and in the setting we will describe later) are finitely generated commutative Hopf algebras.

The following lemma is a generalisation of [EGNO15, Proposition 5.3.6], and the sketch of the proof was inspired by this MathOverflow question ([use17]).

Lemma 6.4.5. *Let $(A, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra in a pre-additive braided monoidal category $\mathcal{C}$ with a bilinear monoidal product. We have*



(6.113)

Sketch of proof. It is easy to see that $S \circ \nabla$ is a left inverse for ∇ with regard the convolution product (using (6.108)). Using graphical calculus it is also not very hard to see that $\nabla \circ S \circ \gamma_{(A,A)}$ is a right inverse for ∇ with regard to this product. As the convolution is associative, this implies that they are equal. ◪

6.4.2 Modules and comodules over Hopf algebras

We are interested in modules and comodules over Hopf algebras. A (co)module over a Hopf algebra is simply a (co)module over the underlying (co)algebra. This means that the algebra or coalgebra structure alone suffices to define (co)modules, and the additional structure of the Hopf algebra does not affect the basic definition. However, we will see that the presence of both structures (and in particular, their compatibility) allows one to endow the category of modules or comodules with a monoidal structure. One could summarise this by saying that the algebra structure determines the abelian structure on the category of modules, while the coalgebra structure is responsible for endowing this category with a monoidal structure.

Modules and comodules over Hopf algebras are interesting because they describe representations, as the following standard examples show.

Example 53 (Representations of groups as modules over the group algebra). Let G be a group, $\mathbb{K}$ a field, and $\mathbb{K}G$ the group algebra of G over $\mathbb{K}$. Modules over $\mathbb{K}G$, viewed as a $\mathbb{K}$-algebra, are precisely $\mathbb{K}$-linear representations of G. However, the tensor product $\otimes_{\mathbb{K}G}$ of $\mathbb{K}G$-modules does not coincide with the tensor product of representations, which is defined using the tensor product $\otimes_{\mathbb{K}}$ of vector spaces. Also, a priori, there is no natural action of G on the tensor product $V \otimes_{\mathbb{K}} W$ of two such representations.

Example 54 (Representations of affine algebraic groups as comodules over the coordinate algebra). Comodules of the coalgebra defined by the coordinate algebra of an affine algebraic group are the representations of this affine algebraic group.

Example 55. Every object $X \in \mathrm{Ob}(\mathcal{C})$ defines a $(\mathbb{1}, \rho_{\mathbb{1}})$-module (X, λ_X) and a $(\mathbb{1}, \rho_{\mathbb{1}}^{-1})$-comodule (X, λ_X^{-1}). This can easily be seen through the triangle identity (3.5).

Definition 6.4.6 (Monoidal products of modules and comodules). Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive braided monoidal category such that the monoidal product is bilinear on morphisms, and let (A, ∇, Δ) be a bialgebra in this category.

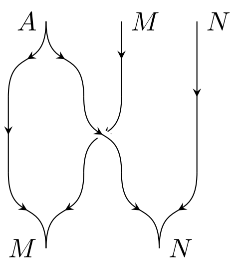
1. Let $(M, \triangleright_M), (N, \triangleright_N)$ be two (left) modules over the algebra (A, ∇). We define the tensor product $(M, \triangleright_M) \otimes_{\mathrm{mod}} (N, \triangleright_N) = (M \otimes N, \triangleright_{M\otimes N})$ as the module given by the object $M \otimes N \in \mathrm{Ob}(\mathcal{C})$, equipped with the action

$$A \otimes M \otimes N \xrightarrow{\Delta\otimes \mathrm{id}_{M\otimes N}} A \otimes A \otimes M \otimes N \xrightarrow{\mathrm{id}_A \otimes \gamma_{(A,M)} \otimes \mathrm{id}_N} A \otimes M \otimes A \otimes N \xrightarrow{\triangleright_M \otimes \triangleright_N} M \otimes N \ , \tag{6.114}$$

or writing out the associators

$$\begin{array}{ccc}
A \otimes (M \otimes N) & \xrightarrow{\triangleright_{M\otimes N}} & M \otimes N \\
\downarrow{\scriptstyle \Delta\otimes \mathrm{id}_{M\otimes N}} & & \uparrow{\scriptstyle \triangleright_M \otimes \triangleright_N} \\
(A \otimes A) \otimes (M \otimes N) & & (A \otimes M) \otimes (A \otimes N) \\
\downarrow{\scriptstyle \alpha_{(A,A,M\otimes N)}} & & \uparrow{\scriptstyle \alpha^{-1}_{(A,M,A\otimes N)}} \\
A \otimes (A \otimes (M \otimes N)) & & A \otimes (M \otimes (A \otimes N)) \\
\downarrow{\scriptstyle \mathrm{id}_A \otimes \alpha^{-1}_{(A,M,N)}} & & \uparrow{\scriptstyle \mathrm{id}_A \otimes \alpha_{(M,A,N)}} \\
A \otimes ((A \otimes M) \otimes N) & \xrightarrow{\mathrm{id}_A \otimes (\gamma_{(A,M)} \otimes \mathrm{id}_N)} & A \otimes ((M \otimes A) \otimes N)
\end{array} \ . \tag{6.115}$$

Graphically, this is

$\triangleright_{M\otimes N} =$  . (6.116)

2. Let $(M, \triangleright_M)$, $(N, \triangleright_N)$ be two (left) comodules over the coalgebra (A, Δ). We define the tensor product $(M, \triangleright_M) \otimes_{\text{comod}} (N, \triangleright_N) = (M \otimes N, \triangleright_{M\otimes N})$ as the comodule given by the object $M \otimes N \in \text{Ob}(\mathcal{C})$, equipped with the coaction

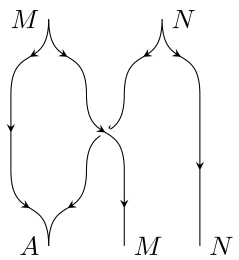
$$M \otimes N \xrightarrow{\triangleright_M \otimes \triangleright_N} A \otimes M \otimes A \otimes N \xrightarrow{\text{id}_A \otimes \gamma_{(M,A)} \otimes \text{id}_N} A \otimes A \otimes M \otimes N \xrightarrow{\nabla \otimes \text{id}_{M\otimes N}} A \otimes M \otimes N \ , \tag{6.117}$$

or writing out the associators

$$\begin{array}{ccc} M \otimes N & \overset{\triangleright_{M\otimes N}}{\dashrightarrow} & A \otimes (M \otimes N) \\ \downarrow{\scriptstyle \triangleright_M \otimes \triangleright_N} & & \uparrow{\scriptstyle \nabla \otimes \text{id}_{M\otimes N}} \\ (A \otimes M) \otimes (A \otimes N) & & (A \otimes A) \otimes (M \otimes N) \\ \downarrow{\scriptstyle \alpha_{(A,M,A\otimes N)}} & & \uparrow{\scriptstyle \alpha^{-1}_{(A,A,M\otimes N)}} \\ A \otimes (M \otimes (A \otimes N)) & & A \otimes (A \otimes (M \otimes N)) \\ \downarrow{\scriptstyle \text{id}_A \otimes \alpha^{-1}_{(M,A,N)}} & & \uparrow{\scriptstyle \text{id}_A \otimes \alpha_{(A,M,N)}} \\ A \otimes ((M \otimes A) \otimes N) & \xrightarrow{\text{id}_A \otimes (\gamma_{(M,A)} \otimes \text{id}_N)} & A \otimes ((A \otimes M) \otimes N) \end{array} \ . \tag{6.118}$$

Graphically, this is

$$\triangleright_{M\otimes N} = \tag{6.119}$$



Example 56. Let $\mathbb{K}$ be a field, let G be a group, and let $\mathbb{K}G$ be the Hopf algebra introduced in Example 51. Let M, N be arbitrary $\mathbb{K}G$-modules, or equivalently $\mathbb{K}$-linear G-representations. For $g \in G$ and $x \in M, y \in N$, Definition 6.4.6 gives

$$g \triangleright (x \otimes y) = (g \triangleright x) \otimes (g \triangleright y). \tag{6.120}$$

This shows that the above monoidal product corresponds to the standard tensor product of representations for group algebras.

We begin by showing that this monoidal product is well-defined, i.e. that the monoidal product of modules is a module.

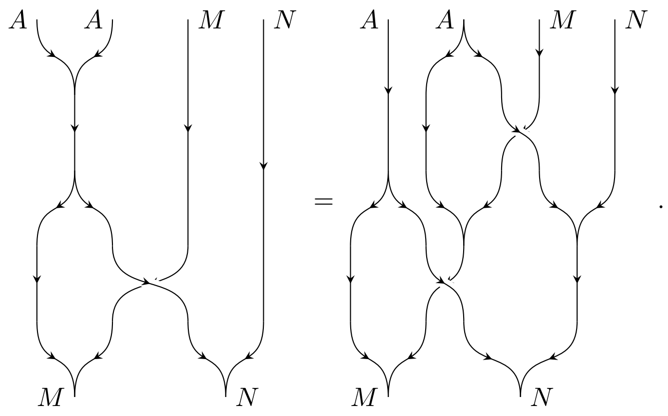
Proposition 6.4.7. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive braided monoidal category such that the monoidal product is bilinear on morphisms, and let (A, ∇, Δ) be an associative bialgebra in this category.*

1. *If $(M, \triangleright_M)$ and $(N, \triangleright_N)$ are two (A, ∇)-modules (resp. (A, Δ)-comodules), then the tensor product $(M \otimes N, \triangleright_{M\otimes N})$ of these two modules (resp. comodules) is again an (A, ∇)-module (resp. an (A, Δ)-comodule).*
2. *If $f : (M, \triangleright_M) \to (N, \triangleright_N)$ and $\overline{f} : (\overline{M}, \triangleright_{\overline{M}}) \to (\overline{N}, \triangleright_{\overline{N}})$ are morphisms between (A, ∇)-modules (resp. (A, Δ)-comodules), then the tensor product $f \otimes \overline{f}$ as morphisms in $\mathcal{C}$ is a morphism $(M \otimes \overline{M}, \triangleright_{M\otimes\overline{M}}) \to (N \otimes \overline{N}, \triangleright_{N\otimes\overline{N}})$.*

Proof. We have to check that the following diagram commutes

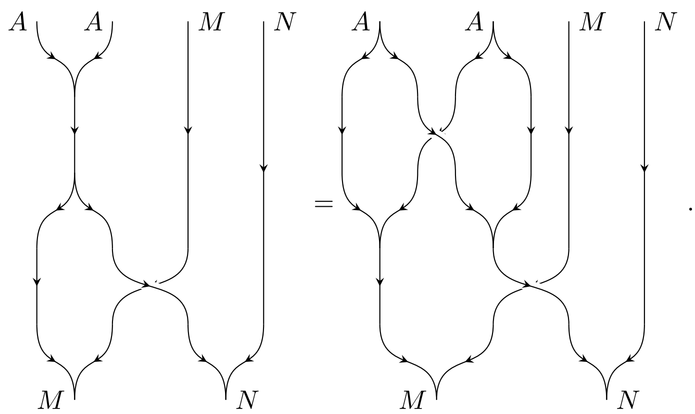
$$\begin{array}{ccc} A \otimes A \otimes M \otimes N & \xrightarrow{\nabla \otimes \text{id}_{M\otimes N}} & A \otimes M \otimes N \\ \downarrow{\scriptstyle \text{id}_A \otimes \triangleright_{M\otimes N}} & & \downarrow{\scriptstyle \triangleright_{M\otimes N}} \\ A \otimes M \otimes N & \xrightarrow{\triangleright_{M\otimes N}} & M \otimes N \end{array} \ , \tag{6.121}$$

or graphically



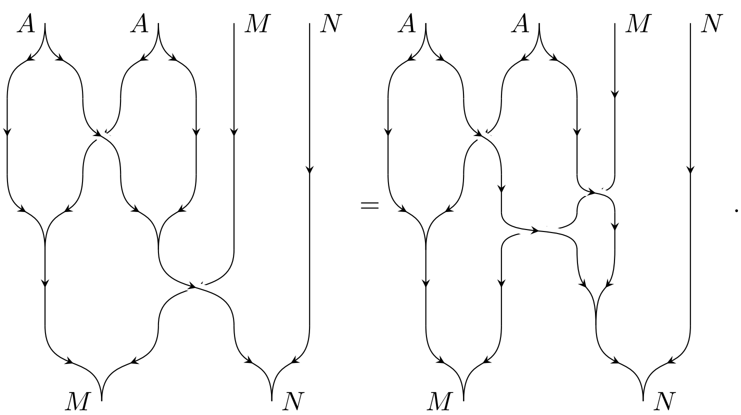
(6.122)

Using the bialgebra condition (6.106), we find



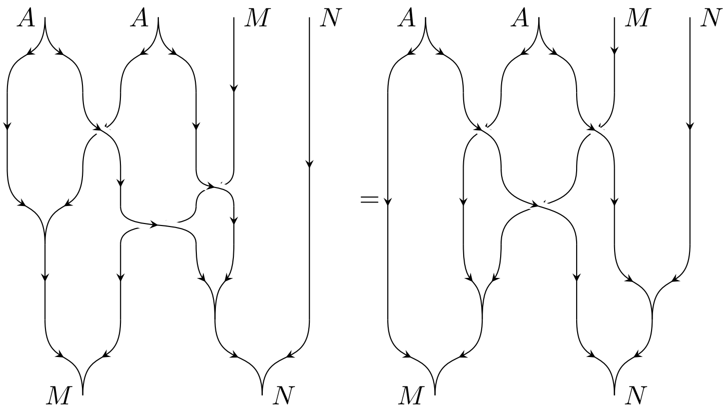
(6.123)

Applying the naturality of the braiding (3.79), and the hexagon identity (3.80), we find



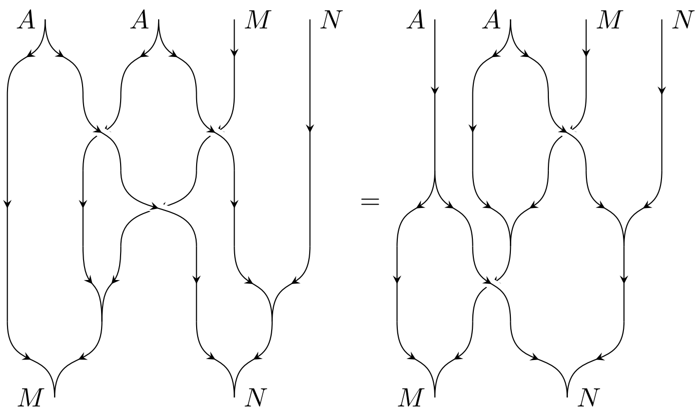
(6.124)

Using that $(M, \triangleright_M)$ and $(N, \triangleright_N)$ are left modules (6.21), we find



. (6.125)

Applying the hexagon identity (3.80) and the naturality of the braiding (3.79) once again, we finally obtain

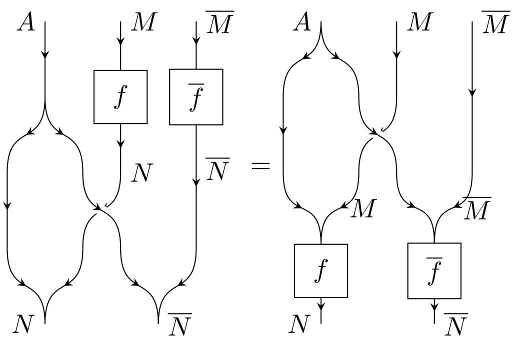


. (6.126)

Suppose now that $f : (M, \triangleright_M) \to (N, \triangleright_N)$ and $\overline{f} : (\overline{M}, \triangleright_{\overline{M}}) \to (\overline{N}, \triangleright_{\overline{N}})$ are morphisms between (A, ∇)-modules. We want to show that $f \otimes \overline{f}$ is an (A, ∇)-module morphism too, i.e. that the following diagram commutes

$$\begin{array}{ccc} A \otimes M \otimes \overline{M} & \xrightarrow{\mathrm{id}_A \otimes f \otimes \overline{f}} & A \otimes N \otimes \overline{N} \\ \downarrow{\scriptstyle \triangleright_{M \otimes \overline{M}}} & & \downarrow{\scriptstyle \triangleright_{N \otimes \overline{N}}} \\ M \otimes \overline{M} & \xrightarrow{f \otimes \overline{f}} & N \otimes \overline{N} \end{array}, \tag{6.127}$$

or equivalently that



. (6.128)

This follows from the naturality of the braiding (3.79), and the fact that f and $\overline{f}$ are module morphisms (6.30). ■

We can now show that this monoidal product endows the category of modules with a monoidal structure.

Proposition 6.4.8. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive (or more generally enriched over ${}_R\mathbf{Mod}$, where R is a commutative ring) braided monoidal category such that the monoidal product is bilinear on morphisms, and let (A, ∇, Δ) be a bialgebra in this category.*

1. *If (A, Δ, ε) is coassociative and counital, then $({}_{(A,\nabla)}\mathbf{Mod}, \otimes_{\mathrm{mod}}, (\mathbb{1}, \varepsilon \circ \rho_A), \alpha, \lambda, \rho)$ is a pre-additive (or ${}_R\mathbf{Mod}$-enriched) monoidal category such that the monoidal product is bilinear on objects. The monoidal product bifunctor $\otimes_{\mathrm{mod}}$ is defined on objects in Definition 6.4.6, and on morphisms in Proposition 6.4.7.*

   If, in addition, the braiding γ is symmetric and (A, Δ) is cocommutative, then the module category is symmetric braided when equipped with the inherited braiding.

2. *If (A, ∇, η) is associative and unital, then $({}_{(A,\Delta)}\mathbf{Comod}, \otimes_{\mathrm{comod}}, (\mathbb{1}, \rho_A^{-1} \circ \eta), \alpha, \lambda, \rho)$ is a pre-additive (or ${}_R\mathbf{Mod}$-enriched) monoidal category such that the monoidal product is bilinear on objects.*

   If, in addition, the braiding γ is symmetric and (A, ∇) is commutative, then the module category is symmetric braided when equipped with the inherited braiding.

Proof. Observe first that the category ${}_{(A,\nabla)}\mathbf{Mod}$ is pre-additive (or, more generally, ${}_R\mathbf{Mod}$-enriched) because the sum of two module morphisms again satisfies the module compatibility condition (6.29), by bilinearity of composition.

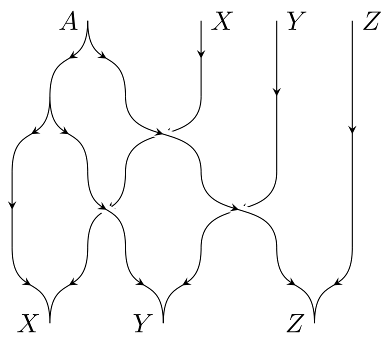
Moreover, as the tensor product of morphisms in ${}_{(A,\nabla)}\mathbf{Mod}$ is inherited from that in $\mathcal{C}$, it follows that $\otimes : {}_{(A,\nabla)}\mathbf{Mod} \times {}_{(A,\nabla)}\mathbf{Mod} \to {}_{(A,\nabla)}\mathbf{Mod}$ is a bifunctor and bilinear on morphisms.

Note that if we can show that the monoidal associator α, the unitors λ and ρ, and the braiding γ are defined in ${}_{(A,\nabla)}\mathbf{Mod}$ (in the sense that their components corresponding to modules are module morphisms), then it follows immediately that these are natural transformations satisfying all the required coherence conditions.

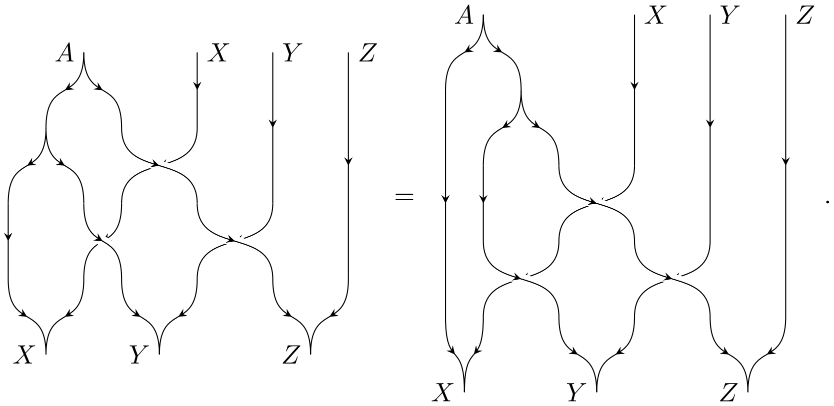
Let $(X, \triangleright_X), (Y, \triangleright_Y), (Z, \triangleright_Z)$ be (A, ∇)-modules, we will prove that $\alpha_{(X,Y,Z)}$ is a module morphism from $((X, \triangleright_X) \otimes_{\mathrm{mod}} (Y, \triangleright_Y)) \otimes_{\mathrm{mod}} (Z, \triangleright_Z)$ to $(X, \triangleright_X) \otimes_{\mathrm{mod}} ((Y, \triangleright_Y) \otimes_{\mathrm{mod}} (Z, \triangleright_Z))$. To prove this, we have to know that the following diagram commutes

$$\begin{array}{ccc} A \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\mathrm{id}_A \otimes \alpha_{(X,Y,Z)}} & A \otimes (X \otimes (Y \otimes Z)) \\ {\scriptstyle \triangleright_{(X \otimes Y) \otimes Z}} \downarrow & & \downarrow {\scriptstyle \triangleright_{X \otimes (Y \otimes Z)}} \\ (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{(X,Y,Z)}} & X \otimes (Y \otimes Z) \end{array} . \tag{6.129}$$

In string diagrams, we have

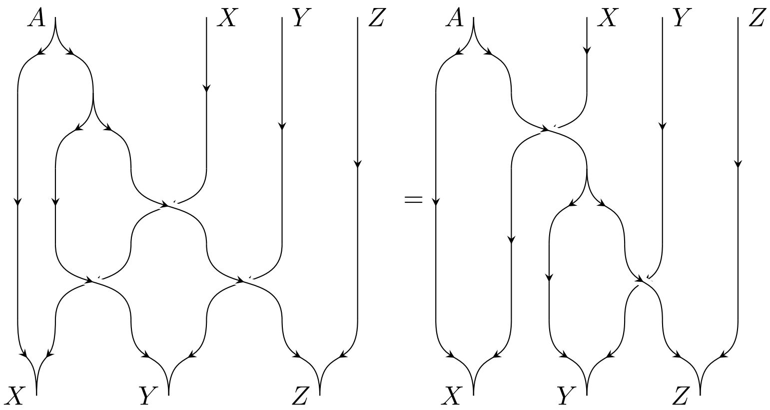
$\triangleright_{(X \otimes Y) \otimes Z} =$  . (6.130)

As (A, Δ) is a coassociative coalgebra (6.9), we have



(6.131)

Using the hexagon identity (3.80) and the naturality of the braiding (3.79), we then find



$$= \triangleright_{X\otimes(Y\otimes Z)}, \tag{6.132}$$

which is what we had to prove.

Next, we will prove that λ_X is a morphism from $(\mathbb{1}, \varepsilon \circ \rho_A) \otimes_{\mathrm{mod}} (X, \triangleright_X)$ to $(X, \triangleright_X)$. To do so, we need to show that the following diagram commutes

$$\begin{array}{ccc} A \otimes (\mathbb{1} \otimes X) & \xrightarrow{\mathrm{id}_A \otimes \lambda_X} & A \otimes X \\ {\scriptstyle \triangleright_{\mathbb{1}\otimes X}} \downarrow & & \downarrow {\scriptstyle \triangleright_X} \\ \mathbb{1} \otimes X & \xrightarrow{\lambda_X} & X \end{array} \; . \tag{6.133}$$

To prove this, we use (3.5), (3.2), (3.1), $\gamma_{(A,\mathbb{1})} = \lambda_A^{-1} \circ \rho_A$ (Lemma 3.6.5), the fact that $(X, \triangleright_X)$ is a left module

(6.19), and the fact that $\varepsilon : A \to \mathbb{1}$ is a counit (6.10), to obtain

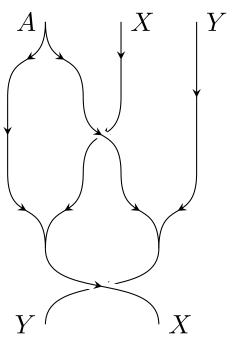
$$\begin{aligned}
&\lambda_X \circ (\triangleright_{\mathbb{1}} \otimes \triangleright_X) \circ \alpha^{-1}_{(A,\mathbb{1},A\otimes X)} \circ (\mathrm{id}_A \otimes \alpha_{(\mathbb{1},A,X)}) \circ (\mathrm{id}_A \otimes (\gamma_{(A,\mathbb{1})} \otimes \mathrm{id}_X)) \\
&\circ (\mathrm{id}_A \otimes \alpha^{-1}_{(A,\mathbb{1},X)}) \circ \alpha_{(A,A,\mathbb{1}\otimes X)} \circ (\Delta \otimes \mathrm{id}_{\mathbb{1}\otimes X}) \\
&= \lambda_X \circ (\varepsilon \otimes \triangleright_X) \circ (\mathrm{id}_A \otimes \lambda_{A\otimes X}) \circ (\mathrm{id}_A \otimes \alpha_{(\mathbb{1},A,X)}) \circ (\mathrm{id}_A \otimes (\gamma_{(A,\mathbb{1})} \otimes \mathrm{id}_X)) \\
&\circ (\mathrm{id}_A \otimes \alpha^{-1}_{(A,\mathbb{1},X)}) \circ \alpha_{(A,A,\mathbb{1}\otimes X)} \circ (\Delta \otimes \mathrm{id}_{\mathbb{1}\otimes X}) \\
&= \lambda_X \circ (\varepsilon \otimes \triangleright_X) \circ (\mathrm{id}_A \otimes (\lambda_A \otimes \mathrm{id}_X)) \circ (\mathrm{id}_A \otimes (\gamma_{(A,\mathbb{1})} \otimes \mathrm{id}_X)) \\
&\circ (\mathrm{id}_A \otimes \alpha^{-1}_{(A,\mathbb{1},X)}) \circ \alpha_{(A,A,\mathbb{1}\otimes X)} \circ (\Delta \otimes \mathrm{id}_{\mathbb{1}\otimes X}) \\
&= \lambda_X \circ (\varepsilon \otimes \triangleright_X) \circ (\mathrm{id}_A \otimes (\rho_A \otimes \mathrm{id}_X)) \circ (\mathrm{id}_A \otimes \alpha^{-1}_{(A,\mathbb{1},X)}) \circ \alpha_{(A,A,\mathbb{1}\otimes X)} \\
&= \lambda_X \circ (\varepsilon \otimes \triangleright_X) \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \otimes \lambda_X)) \circ \alpha_{(A,A,\mathbb{1}\otimes X)} \circ (\Delta \otimes \mathrm{id}_{\mathbb{1}\otimes X}) \\
&= \lambda_X \circ (\varepsilon \otimes \triangleright_X) \circ \alpha_{(A,A,X)} \circ (\mathrm{id}_{A\otimes A} \otimes \lambda_X) \circ (\Delta \otimes \mathrm{id}_{\mathbb{1}\otimes X}) \\
&= \lambda_X \circ (\varepsilon \otimes \triangleright_X) \circ \alpha_{(A,A,X)} \circ (\Delta \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \lambda_X) \\
&= \triangleright_X \circ \lambda_{A\otimes X} \circ (\varepsilon \otimes \mathrm{id}_{A\otimes X}) \circ \alpha_{(A,A,X)} \circ (\Delta \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \lambda_X) \\
&= \triangleright_X \circ \lambda_{A\otimes X} \circ \alpha_{(\mathbb{1},A,X)} \circ ((\varepsilon \otimes \mathrm{id}_A) \otimes \mathrm{id}_X) \circ (\Delta \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \lambda_X) \\
&= \triangleright_X \circ \lambda_{A\otimes X} \circ \alpha_{(\mathbb{1},A,X)} \circ (\lambda_A^{-1} \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \lambda_X) \\
&= \triangleright_X \circ (\mathrm{id}_A \otimes \lambda_X)
\end{aligned} \quad . \tag{6.134}$$

Similarly, we prove that ρ_X is a morphism from $(X, \triangleright_X) \otimes_{\mathrm{mod}} (\mathbb{1}, \varepsilon \circ \rho_A)$ to $(X, \triangleright_X)$.

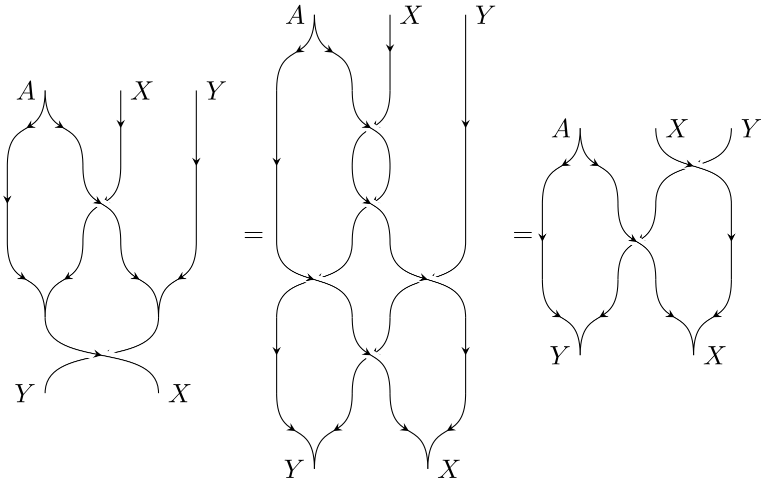
Finally, we want to prove that $\gamma_{(X,Y)}$ is a morphism from $(X, \triangleright_X) \otimes_{\mathrm{mod}} (Y, \triangleright_Y)$ to $(Y, \triangleright_Y) \otimes_{\mathrm{mod}} (X, \triangleright_X)$ if γ is symmetric and (A, Δ) is cocommutative. This translates to the following commutative diagram

$$\begin{array}{ccc}
A \otimes (X \otimes Y) & \xrightarrow{\mathrm{id}_A \otimes \gamma_{(X,Y)}} & A \otimes (Y \otimes X) \\
{\scriptstyle \triangleright_{X\otimes Y}} \big\downarrow & & \big\downarrow {\scriptstyle \triangleright_{Y\otimes X}} \\
X \otimes Y & \xrightarrow{\quad \gamma_{(X,Y)} \quad} & Y \otimes X
\end{array} \quad . \tag{6.135}$$

Graphically, we have

$\gamma_{(X,Y)} \circ \triangleright_{X\otimes Y} =$  . (6.136)

Naturality of the braiding (3.79), the hexagon identity (3.80), symmetry of the braiding, and cocommutativity of (A,Δ) give



$$= \triangleright_{Y\otimes X} \circ (\mathrm{id}_A \otimes \gamma_{(X,Y)}). \quad (6.137)$$

■

We are interested in symmetric tensor categories, so we will determine whether module categories over Hopf algebras in symmetric tensor categories are themselves symmetric tensor categories.

Proposition 6.4.9. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \gamma)$ be a pre-additive (or ${}_R\mathbf{Mod}$-enriched, with R some commutative ring) braided monoidal category such that the monoidal product is bilinear on morphisms, and let (A, ∇, Δ) be a bialgebra in this category.*

1. *Suppose that (A, Δ, ε) is coassociative and counital. We will discuss the category ${}_{(A,\nabla)}\mathbf{Mod}$ equipped with the pre-additive (or ${}_R\mathbf{Mod}$-enriched) monoidal structure introduced above.*

   a) *If $\mathcal{C}$ is additive, then ${}_{(A,\nabla)}\mathbf{Mod}$ is additive.*

   b) *If $\mathcal{C}$ is pre-abelian, then ${}_{(A,\nabla)}\mathbf{Mod}$ is pre-abelian.*

   c) *If $\mathcal{C}$ is abelian, then ${}_{(A,\nabla)}\mathbf{Mod}$ is abelian. Furthermore, if the monoidal product is biexact on $\mathcal{C}$, then it is biexact on ${}_{(A,\nabla)}\mathbf{Mod}$.*

   d) *If $\mathcal{C}$ is left rigid (equivalently, right rigid or rigid through Proposition 3.6.7) and $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra with an invertible antipode, then ${}_{(A,\nabla)}\mathbf{Mod}$ is left rigid. If S is invertible, then ${}_{(A,\nabla)}\mathbf{Mod}$ is rigid.*

   e) *If $\mathcal{C}$ is a (multi)tensor category and $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra, then ${}_{(A,\nabla)}\mathbf{Mod}$ is a (multi)tensor category.*

   f) *If $\mathcal{C}$ is a symmetric (multi)tensor category and $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra with an invertible antipode, then ${}_{(A,\nabla)}\mathbf{Mod}$ is a symmetric (multi)tensor category.*

2. *Suppose that (A, ∇, η) is associative and unital. We will discuss the category ${}_{(A,\Delta)}\mathbf{Comod}$ equipped with the pre-additive (or ${}_R\mathbf{Mod}$-enriched) monoidal structure introduced above.*

   a) *If $\mathcal{C}$ is additive, then ${}_{(A,\Delta)}\mathbf{Comod}$ is additive.*

   b) *If $\mathcal{C}$ is pre-abelian, then ${}_{(A,\Delta)}\mathbf{Comod}$ is pre-abelian.*

   c) *If $\mathcal{C}$ is abelian, then ${}_{(A,\Delta)}\mathbf{Comod}$ is abelian. Furthermore, if the monoidal product is biexact on $\mathcal{C}$, then it is biexact on ${}_{(A,\Delta)}\mathbf{Comod}$.*

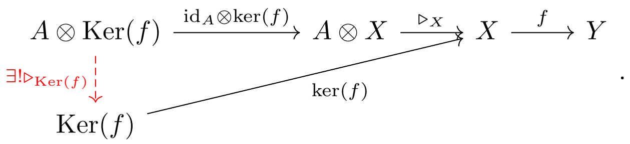
   d) *If $\mathcal{C}$ is rigid (equivalently, left or right rigid) and $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra, then ${}_{(A,\Delta)}\mathbf{Comod}$ is left rigid. If S is invertible, then ${}_{(A,\Delta)}\mathbf{Comod}$ is rigid.*

   e) *If $\mathcal{C}$ is a (multi)tensor category and $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra with an invertible antipode, then ${}_{(A,\Delta)}\mathbf{Comod}$ is a (multi)tensor category.*

f) If $\mathcal{C}$ is a symmetric (multi)tensor category and $(A, \nabla, \eta, \Delta, \varepsilon, S)$ is a commutative Hopf algebra with an invertible antipode, then ${}_{(A,\Delta)}\mathbf{Comod}$ is a symmetric (multi)tensor category.

Proof. For (1a), we need to show that biproducts of modules are modules and that the null object is a module. Both statements follow from the bilinearity of the monoidal product $\otimes$ on morphisms (Proposition 4.2.1).

For (1b), we have to prove that kernels and cokernels of module morphisms are modules, and that the kernel and cokernel morphisms are morphism of modules. We will prove this for kernels. Let $f : (X, \triangleright_X) \to (Y, \triangleright_Y)$ be a morphism of modules. In $\mathcal{C}$, this morphism has a kernel $(\mathrm{Ker}(f), \ker(f))$. We then have $f \circ \triangleright_X \circ (\mathrm{id}_A \otimes \ker(f)) = \triangleright_Y \circ (\mathrm{id}_A \otimes f) \circ (\mathrm{id}_A \otimes \ker(f)) = 0$, which implies that there is a uniquely induced morphism $\triangleright_{\mathrm{Ker}(f)} : A \otimes \mathrm{Ker}(f) \to \mathrm{Ker}(f)$ making the following diagram commute



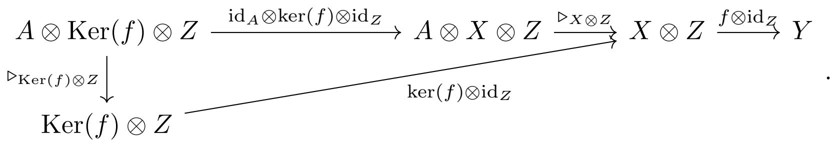
. (6.138)

We now have to prove that $(\mathrm{Ker}(f), \triangleright_{\mathrm{Ker}(f)})$ is a left module, i.e. that $\triangleright_{\mathrm{Ker}(f)} \circ (\mathrm{id}_A \otimes \triangleright_{\mathrm{Ker}(f)}) = \triangleright_{\mathrm{Ker}(f)} \circ (\nabla \otimes \mathrm{id}_{\mathrm{Ker}(f)})$. As $(X, \triangleright_X)$ is a left module, we find

$$\begin{aligned} \ker(f) \circ \triangleright_{\mathrm{Ker}(f)} \circ (\nabla \otimes \mathrm{id}_{\mathrm{Ker}(f)}) &= \triangleright_X \circ (\mathrm{id}_A \otimes \ker(f)) \circ (\nabla \otimes \mathrm{id}_{\mathrm{Ker}(f)}) \\ &= \triangleright_X \circ (\nabla \otimes \mathrm{id}_X) \circ (\mathrm{id}_{A\otimes A} \otimes \ker(f)) \\ &= \triangleright_X \circ (\mathrm{id}_A \otimes \triangleright_X) \circ (\mathrm{id}_{A\otimes A} \otimes \ker(f)) . \\ &= \triangleright_X \circ (\mathrm{id}_A \otimes (\ker(f) \circ \triangleright_{\mathrm{Ker}(f)})) \\ &= \ker(f) \circ \triangleright_{\mathrm{Ker}(f)} \circ (\mathrm{id}_A \otimes \triangleright_{\mathrm{Ker}(f)}) \end{aligned} \tag{6.139}$$

As $\ker(f)$ is a monomorphism, we conclude that $(\mathrm{Ker}(f), \triangleright_{\mathrm{Ker}(f)})$ is indeed a module. The commutativity of (6.138) also implies that $\ker(f)$ is a module morphism $(\mathrm{Ker}(f), \triangleright_{\mathrm{Ker}(f)}) \to (X, \triangleright_X)$. Combining these two facts shows that $((\mathrm{Ker}(f), \triangleright_{\mathrm{Ker}(f)}), \ker(f))$ is a kernel of f in the category of modules.

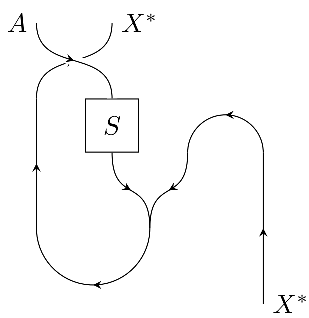
The first statement of (1c) follows from (1b). Suppose that $\otimes$ is biexact in $\mathcal{C}$, and let $(Z, \triangleright_Z)$ be an arbitrary module. As $\otimes$ is biexact, we know that $\mathrm{Ker}(f \otimes \mathrm{id}_Z) = \mathrm{Ker}(f) \otimes Z$ and $\ker(f) \otimes \mathrm{id}_Z = \ker(f \otimes \mathrm{id}_Z)$. All that is left for us to prove that $(\mathrm{Ker}(f) \otimes Z, \triangleright_{\mathrm{Ker}(f)\otimes Z})$ is the kernel of $f \otimes \mathrm{id}_Z$, is then that $\triangleright_{\mathrm{Ker}(f)\otimes Z}$ makes the following diagram commute



. (6.140)

This follows from the fact that $\ker(f)$ is a morphism of modules, which implies that $\ker(f) \otimes \mathrm{id}_Z$ is also a morphism of modules, by Proposition 6.4.8. Similarly, we show that the monoidal product $\otimes_{\mathrm{mod}}$ preserves kernels in the second argument, as well as cokernels in both the first and second argument. We conclude that $\otimes_{\mathrm{mod}}$ is biexact.

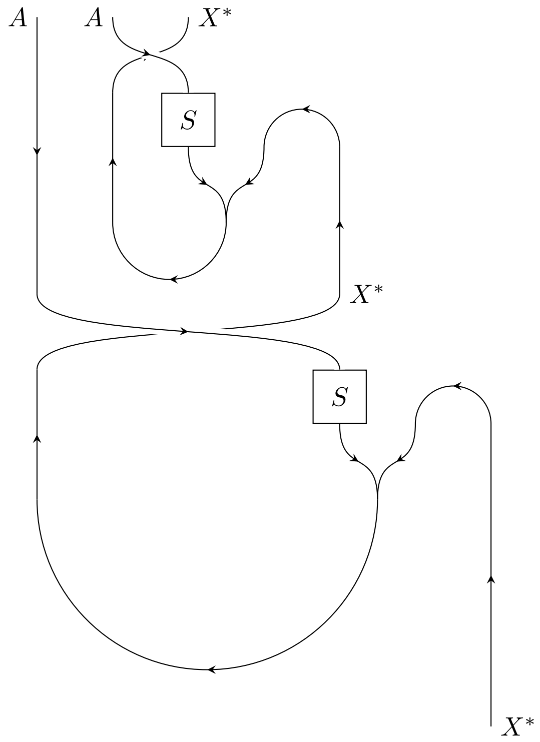
We will now prove (1d), so suppose that $\mathcal{C}$ is left rigid (equivalently, rigid or right rigid) and that there exists an antipode $S : A \to A$ which is such that $S \circ \nabla = \nabla \circ (S \otimes S) \circ \gamma_{(A,A)}$ through Lemma 6.4.5. We define the left dual of a module $(X, \triangleright_X)$ as the object X^*, equipped with the action

$\triangleright_{X^*} =$


. (6.141)

First, we have to prove that $(X^*, \triangleright_{X^*})$ is a module. We have

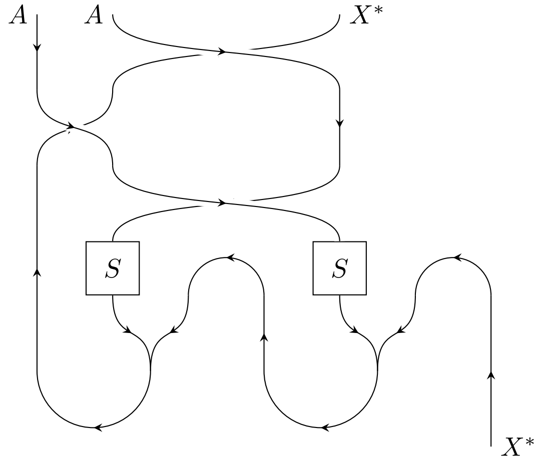
$$\triangleright_{X^*} \circ (\mathrm{id}_A \otimes \triangleright_{X^*}) =$$



$$. \tag{6.142}$$

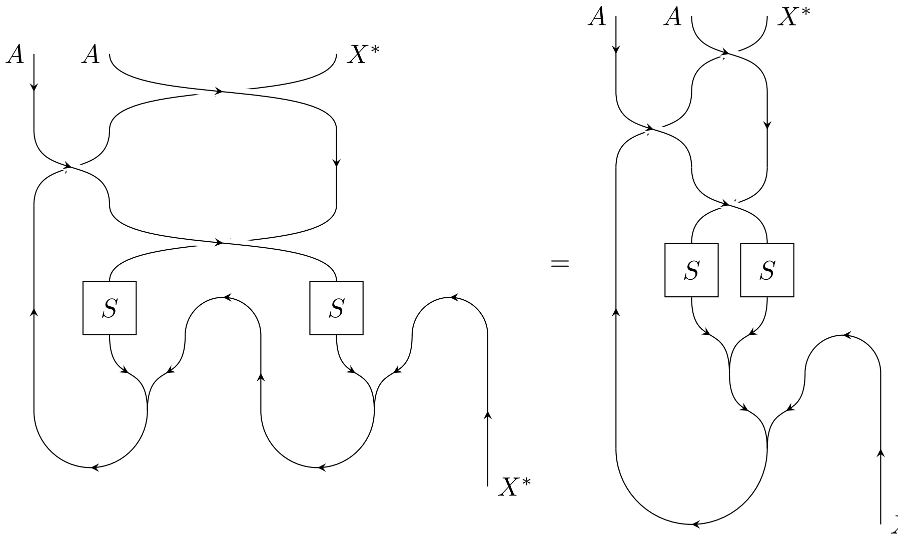
Applying the naturality of the braiding (3.79) and the hexagon identity (3.77), we obtain

$$\triangleright_{X^*} \circ (\mathrm{id}_A \otimes \triangleright_{X^*}) =$$



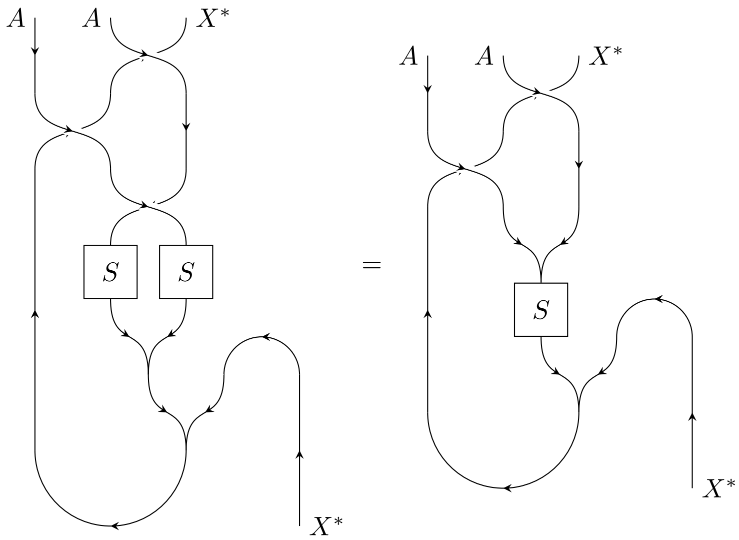
$$. \tag{6.143}$$

Using the zigzag identity (3.18) and the module identity (6.21), we obtain



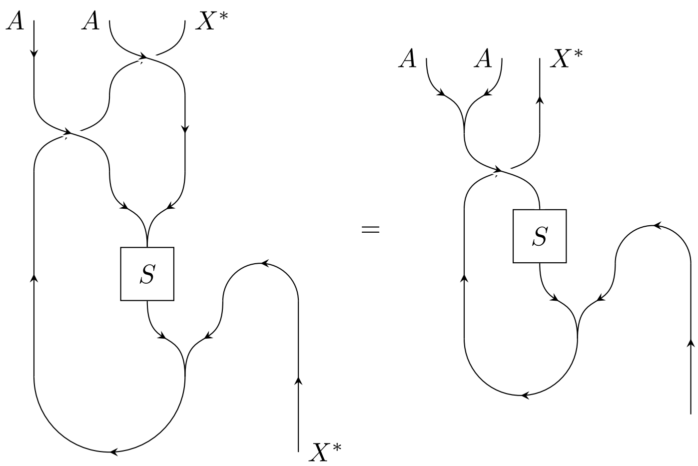
. (6.144)

Applying $\nabla \circ (S \otimes S) \circ \gamma_{(A,A)} = S \circ \nabla$, we see that



. (6.145)

Applying the naturality of the braiding (3.79), we finally get



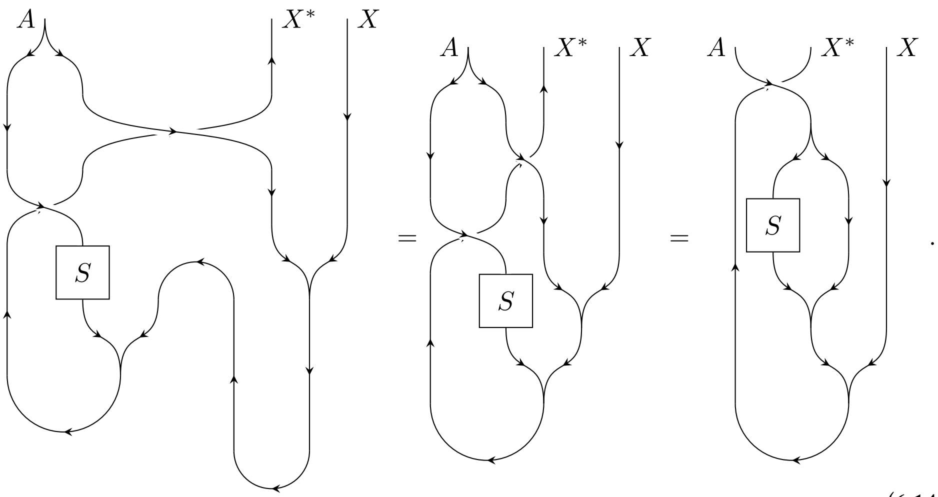
$$= \triangleright_{X^*} \circ (\nabla \otimes \mathrm{id}_{X^*}). \quad (6.146)$$

To conclude that $(X^*, \triangleright_{X^*})$ is a left dual of $(X, \triangleright_X)$, we also have to show that ev_X and coev_X are morphisms in ${}_{(A,\nabla)}\mathbf{Mod}$. We thus have to prove that the following diagrams commute

$$\begin{array}{ccc} A \otimes (X^* \otimes X) & \xrightarrow{\mathrm{id}_A \otimes \mathrm{ev}_X} & A \otimes \mathbb{1} \\ {\scriptstyle \triangleright_{X^* \otimes X}} \downarrow & & \downarrow {\scriptstyle \triangleright_{\mathbb{1}}} \\ X^* \otimes X & \xrightarrow{\mathrm{ev}_X} & \mathbb{1} \end{array} \qquad \begin{array}{ccc} A \otimes \mathbb{1} & \xrightarrow{\mathrm{id}_A \otimes \mathrm{coev}_X} & A \otimes (X \otimes X^*) \\ {\scriptstyle \triangleright_{\mathbb{1}}} \downarrow & & \downarrow {\scriptstyle \triangleright_{X \otimes X^*}} \\ \mathbb{1} & \xrightarrow{\mathrm{coev}_X} & X \otimes X^* \end{array} . \tag{6.147}$$

We will only prove the commutativity of the first of those two diagrams. We have

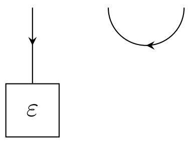
$$\mathrm{ev}_X \circ \triangleright_{X^* \otimes X} =$$



(6.148)

Using the antipode identity (6.111), and the fact that η is a unit for (A, ∇), this becomes

$$\mathrm{ev}_X \circ \triangleright_{X^* \otimes X} =$$



$$= \triangleright_{\mathbb{1}} \circ (\mathrm{id}_A \otimes \mathrm{ev}_X). \tag{6.149}$$

This proves (1d).

Finally, (1e) and (1f) follow from the other statements and Proposition 6.4.8. ■

6.5 Affine group schemes in categories

Using the theory of algebras built up in this chapter, we can define affine algebraic groups as in [Med25]. We will provide only the most basic definitions, as these, combined with our discussion on categories of modules and comodules over Hopf algebras, suffice to understand the statements given in Section 4.5.3.

Definition 6.5.1 (Affine group schemes and affine algebraic groups, [Med25, Definition 5.1.1]). Let $\mathcal{C}$ be a symmetric tensor category. An *affine group scheme* in $\mathcal{C}$ or $\mathcal{C}^{\mathrm{ind}}$ is a functor

$$G : \mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}} \to \mathbf{Grp} \text{ such that } \mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}} \circ G \text{ is representable.} \tag{6.150}$$

If G is representable by a finitely generated algebra, then G is called an *affine group scheme of finite type*, or an *affine algebraic group*.

The representing algebra (unique up to isomorphism) is called the *coordinate algebra* and is denoted $\mathcal{O}[G]$. This algebra can be equipped with the structure of a commutative Hopf algebra in the standard way: we have natural transformations

1. $\mathrm{mult} : G \times G \to G$ where

$$\mathrm{mult}_{(A,\mu,\eta)} \text{ is the product on the group } G(A,\mu,\eta), \tag{6.151}$$

using the coordinate algebra we can interpret this as a natural transformation

$$\mathrm{mult} : \mathrm{Hom}_{\mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}}}\left(\mathcal{O}[G] \otimes \mathcal{O}[G], -\right) \to \mathrm{Hom}_{\mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}}}\left(\mathcal{O}[G], -\right), \tag{6.152}$$

hence as a morphism

$$\Delta : \mathcal{O}[G] \to \mathcal{O}[G] \otimes \mathcal{O}[G] \tag{6.153}$$

through the Yoneda lemma 1.3.1,

2. $\mathrm{inv} : G \to G$ where

$$\mathrm{inv}_{(A,\mu,\eta)} \text{ is the inversion on the group } G(A,\mu,\eta), \tag{6.154}$$

using the coordinate algebra we can interpret this as a natural transformation

$$\mathrm{inv} : \mathrm{Hom}_{\mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}}}\left(\mathcal{O}[G], -\right) \to \mathrm{Hom}_{\mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}}}\left(\mathcal{O}[G], -\right), \tag{6.155}$$

hence as a morphism

$$S : \mathcal{O}[G] \to \mathcal{O}[G], \tag{6.156}$$

3. $\mathrm{unit} : 1 \to G$ where

$$\mathrm{unit}_{(A,\mu,\eta)} \text{ maps the trivial group to the unit of the group } G(A,\mu,\eta), \tag{6.157}$$

using the coordinate algebra we can intepret this as a natural transformation

$$\mathrm{unit} : \mathrm{Hom}_{\mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}}}(\mathbb{1}, -) \to \mathrm{Hom}_{\mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}}}\left(\mathcal{O}[G], -\right), \tag{6.158}$$

hence as a morphism

$$\varepsilon : \mathbb{1} \to \mathcal{O}[G]. \tag{6.159}$$

Lemma 6.5.2 ([Med25, Lemma 5.1.5]). *Let $\mathcal{C}$ be a symmetric tensor category. A functor*

$$F : \mathbf{AssCommUnitAlg}_{\mathcal{C}^{\mathrm{ind}}} \to \mathbf{Set}$$

is the composition of $\mathrm{Forgetful}^{\mathbf{Set}}_{\mathbf{Grp}}$ *with an affine group scheme G if and only if there exist natural transformations*

$$\mathrm{mult} : G \times G \to G, \tag{6.160}$$
$$\mathrm{inv} : G \to G, \tag{6.161}$$
$$\mathrm{unit} : 1 \to G, \tag{6.162}$$

such that the following diagrams commute

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}_G \times \mathrm{mult}} & G \times G \\ \downarrow{\scriptstyle \mathrm{mult} \times \mathrm{id}_G} & & \downarrow{\scriptstyle \mathrm{mult}} \\ G \times G & \xrightarrow{\mathrm{mult}} & G \end{array}, \tag{6.163}$$

$$\begin{array}{ccc} G \times 1 & \xrightarrow{\mathrm{id}_G \times \mathrm{unit}} & G \times G \\ \downarrow & & \downarrow{\scriptstyle \mathrm{mult}} \\ G & \xrightarrow{\mathrm{id}_G} & G \end{array} \qquad \begin{array}{ccc} 1 \times G & \xrightarrow{\mathrm{unit} \times \mathrm{id}_G} & G \times G \\ \downarrow & & \downarrow{\scriptstyle \mathrm{mult}} \\ G & \xrightarrow{\mathrm{id}_G} & G \end{array}, \tag{6.164}$$

$$\begin{array}{ccccc} G & \xrightarrow{\mathrm{diag}} & G \times G & \xrightarrow{\mathrm{id}_G \times \mathrm{inv}} & G \times G \\ \downarrow & & & & \downarrow{\scriptstyle \mathrm{mult}} \\ 1 & & \xrightarrow{\mathrm{unit}} & & G \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\mathrm{diag}} & G \times G & \xrightarrow{\mathrm{inv} \times \mathrm{id}_G} & G \times G \\ \downarrow & & & & \downarrow{\scriptstyle \mathrm{mult}} \\ 1 & & \xrightarrow{\mathrm{unit}} & & G \end{array}. \tag{6.165}$$

Proof. This follows by evaluating on objects and realising that the commutative diagrams simply express the group axioms. ■

Proposition 6.5.3 ([Med25, Proposition 5.1.6]). *Let $\mathcal{C}$ be a symmetric tensor category, and let A be a finitely generated commutative unital associative algebra in $\mathcal{C}$. Then A is the coordinate algebra of an affine algebraic group if and only if it admits a Hopf algebra structure. Moreover, for any affine algebraic group, the associated coordinate algebra carries the Hopf algebra structure described above.*

Proof. This follows from the above Lemma 6.5.2 and the contravariant Yoneda embedding 1.3.3 (i.e. the Yoneda lemma). ■

A natural question is how much more general affine group schemes or affine algebraic groups can be in symmetric tensor categories.

In the final chapter, we mention a result by Siddharth Venkatesh [Ven23, Theorem 1.2], which shows that affine algebraic groups in the important Verlinde category are only as general as the Lie algebras they correspond to. That is, their additional generality precisely matches that of Lie algebras in this category.

7 The Verlinde category

In this chapter we discuss the (universal) Verlinde category over an algebraically closed field of characteristic $p > 0$, first introduced in [GK92; GM94]. Our discussion of this category is based on the discussions in [Ost20; Kan24].

There are a few equivalent definitions of the Verlinde category, but the common ground between these definitions is that they describe the category as the semisimplification of a category of modules.

7.1 Representations of the linear algebraic group α_p

We will construct the Verlinde category as the semisimplification of the category of representations of the affine (linear) algebraic group α_p. Proposition 6.4.9 shows that this is a symmetric tensor category.

The discussion in this section is based on the description of the Verlinde category in [Kan24, §3.2].

7.1.1 Description of the linear algebraic group α_p and its coordinate algebra

Definition 7.1.1 (The affine algebraic group α_p, [Med25, Examples 5.1.2 (7)]). Let $\mathbb{K}$ be a field of characteristic $p > 0$. The affine algebraic group α_p is defined as

$$\alpha_p : \mathbf{AssCommUnitAlg}_{\mathbb{K}} \to \mathbf{Grp} : (A, +, \cdot) \mapsto (\{a \in A \mid a^p = 0\}, +). \tag{7.1}$$

The coordinate algebra of this affine algebraic group (we refer to [Med25]) is $\mathcal{O}[\alpha_p] = \mathbb{K}[\alpha_p] = \mathbb{K}[t]/(t^p)$, with

1. the standard polynomial multiplication
$$\nabla : \mathbb{K}[t]/(t^p) \otimes \mathbb{K}[t]/(t^p) \to \mathbb{K}[t]/(t^p) : t^m \otimes t^n \mapsto t^{m+n}, \tag{7.2}$$
2. the standard unit
$$\eta : \mathbb{K} \to \mathbb{K}[t]/(t^p) : 1 \mapsto 1, \tag{7.3}$$
3. the comultiplication
$$\Delta : \mathbb{K}[t]/(t^p) \to \mathbb{K}[t]/(t^p) \otimes \mathbb{K}[t]/(t^p) : t^n \mapsto \sum_{k=0}^{n} \binom{n}{k} t^k \otimes t^{n-k} = (1 \otimes t + t \otimes 1)^n, \tag{7.4}$$
4. the counit
$$\varepsilon : \mathbb{K}[t]/(t^p) \to \mathbb{K} : 1 \mapsto 1 \text{ and } t \mapsto 0, \tag{7.5}$$
5. the antipode
$$S : \mathbb{K}[t]/(t^p) \to \mathbb{K}[t]/(t^p) : t \mapsto -t. \tag{7.6}$$

It is clear that this is a commutative and cocommutative Hopf algebra.

We want to study the category of representations of α_p, $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$, or equivalently the category of finite-dimensional comodules on the Hopf algebra $\mathcal{O}\left[\alpha_p\right] = \mathbb{K}[\alpha_p] = \mathbb{K}[t]/(t^p)$, ${}_{\mathbb{K}[\alpha_p]}\mathbf{FinComod}$. Comodules on a Hopf algebra are equivalent to modules on the dual of this Hopf algebra, so we will work with the category of modules $\mathbb{K}[\alpha_p]^*$, ${}_{\mathbb{K}[\alpha_p]^*}\mathbf{FinMod}$.

We will now describe explicitly what the Hopf algebra structure on $\mathbb{K}[\alpha_p]^*$ is. As a vector space, we have $\mathbb{K}[\alpha_p]^* = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}\big(\mathbb{K}[\alpha_p], \mathbb{K}\big)$, with a basis $\{f_n \mid n = 0, \dots, p-1\}$ such that $f_i(t^j) = \delta_{ij}$ for $i, j \in \{0, \dots, p-1\}$. For the operations on the dual we find

1. the multiplication on $\mathbb{K}[\alpha_p]^*$ is the dual of the comultiplication on $\mathbb{K}[\alpha_p]$
$$\Delta^* = - \circ \Delta : \mathbb{K}[\alpha_p]^* \otimes \mathbb{K}[\alpha_p]^* \to \mathbb{K}[\alpha_p]^* : f_m \otimes f_n \mapsto \binom{m+n}{n} f_{m+n}, \tag{7.7}$$
2. the unit on $\mathbb{K}[\alpha_p]^*$ is given by the dual of the counit on $\mathbb{K}[\alpha_p]$
$$\varepsilon^* = - \circ \varepsilon : \mathbb{K} \to \mathbb{K}[\alpha_p]^* : 1 \mapsto f_0, \tag{7.8}$$
3. the comultiplication on $\mathbb{K}[\alpha_p]^*$ is given by the dual of the multiplication on $\mathbb{K}[\alpha_p]$
$$\nabla^* = - \circ \nabla : \mathbb{K}[\alpha_p]^* \to \mathbb{K}[\alpha_p]^* \otimes \mathbb{K}[\alpha_p]^* : f_n \mapsto \sum_{k=0}^{n} f_k \otimes f_{n-k}, \tag{7.9}$$
4. the counit on $\mathbb{K}[\alpha_p]^*$ is given by the dual of the unit on $\mathbb{K}[\alpha_p]$
$$\eta^* = - \circ \eta : \mathbb{K}[\alpha_p]^* \to \mathbb{K} : f_0 \mapsto 1 \text{ and } f_n \mapsto 0 \text{ for } n \geq 1, \tag{7.10}$$
5. the antipode on $\mathbb{K}[\alpha_p]^*$ is given by the dual of the antipode on $\mathbb{K}[\alpha_p]$
$$S^* = - \circ S : \mathbb{K}[\alpha_p]^* \to \mathbb{K}[\alpha_p]^* : f_n \mapsto -f_n. \tag{7.11}$$

This means that, after rescaling to the basis $g_n := n! f_n$, we find the original Hopf algebra $\mathbb{K}[\alpha_p]$ through the isomorphism $g_n \mapsto t^n$.

We can thus conclude that comodules and modules over $\mathcal{O}\left[\alpha_p\right] = \mathbb{K}[\alpha_p]$ are the same thing.

7.1.2 The abelian structure on the category of modules over $\mathbb{K}[\alpha_p] = \mathbb{K}[t]/(t^p)$

As an abelian category, $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$ is simply the category of $\mathbb{K}[t]/(t^p)$-modules, where $\mathbb{K}[t]/(t^p)$ is considered solely as an algebra with the usual polynomial multiplication.

Lemma 7.1.2. *Let $\mathbb{K}$ be an algebraically closed field. The indecomposable modules in ${}_{\mathbb{K}[t]/(t^p)}\mathbf{FinMod}$ are the modules*
$$J_k := \mathbb{K}[t]/(t^k) \text{ equipped with the polynomial product, for } k = 1, \dots, p. \tag{7.12}$$

Proof. As $\mathbb{K}$ is an algebraically closed field, we know that any $\mathbb{K}[t]$-module decomposes as $\mathbb{K}[t]/(t-\lambda_1)^{k_1} \oplus \cdots \oplus \mathbb{K}[t]/(t-\lambda_n)^{k_n}$ with $k_i \geq 1$. This implies that any $\mathbb{K}[t]/(t^p)$-module decomposes as $\mathbb{K}[t]/(t^{k_1}) \oplus \cdots \oplus \mathbb{K}[t]/(t^{k_n})$ with $1 \leq t \leq p$. Indeed, $\mathbb{K}[t]/(t-\lambda)^k$ is a $\mathbb{K}[t]/(t^p)$-module if and only if t^p acting on this module is zero, i.e. if $t^p \in ((t-\lambda)^k)$.

The modules $J_k = \mathbb{K}[t]/(t^k)$ are indecomposable as they are generated by a single element. ■

Corollary 5.5.4 implies that J_p gets mapped to a null object under the semisimplification functor (because the characteristic of the field is p), and that this is the only indecomposable module that gets mapped to a null object.

7.1.3 The monoidal structure on the category of modules over $\mathbb{K}[\alpha_p] = \mathbb{K}[t]/(t^p)$

Because $\mathbb{K}[\alpha_p]$ is a bialgebra, Definition 6.4.6 defines a tensor product on the category of modules. For our Hopf algebra this results in

$$t \triangleright (x \otimes y) = (1 \otimes t + t \otimes 1) \cdot (x \otimes y) = x \otimes ty + tx \otimes y. \tag{7.13}$$

Since the monoidal product is bilinear on morphisms, it suffices to determine the tensor products of indecomposable objects in order to compute general tensor products.

Lemma 7.1.3. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Let $J_k = \mathbb{K}[t]/(t^k)$ be the indecomposable modules of the category ${}_{\mathbb{K}[t]/(t^p)}\mathbf{FinMod}$ equipped with the tensor produced induced by the Hopf algebra structure on $\mathbb{K}[\alpha_p] = \mathbb{K}[t]/(t^p)$. J_1 is the monoidal unit, and we have*

$$J_k \otimes J_\ell = \bigoplus_{i=1}^{\min(k,\ell,p-k,p-\ell)} J_{|k-\ell|+2i-1} \oplus n_{k\ell} J_p \tag{7.14}$$

for some $n_{k\ell} \geq 0$.

Furthermore, these indecomposable objects are self-dual, i.e. ${J_k}^* \cong J_k$.

Proof. Proposition 6.4.8 shows that the monoidal unit for this tensor product is $(\mathbb{K}, \varepsilon \circ \rho_{\mathbb{K}[\alpha_p]})$. This implies that

$$t \triangleright \lambda = (\varepsilon \circ \rho_{\mathbb{K}[\alpha_p]})(t \otimes \lambda) = \varepsilon(\lambda t) = 0. \tag{7.15}$$

We conclude that J_1 is the unit object for this tensor product.

We will now prove that $J_2 \otimes J_k = J_{k-1} \oplus J_{k+1}$ for $1 < k < p$ and $J_2 \otimes J_p = J_p \oplus J_p$.

The second statement is actually rather simple to prove. We know that J_p gets mapped to a null object under the semisimplification functor. As the semisimplification functor is monoidal, this implies that $J_2 \otimes J_p$ gets mapped to a null object. This implies that the decomposition of $J_2 \otimes J_p$ into indecomposable module can only consist of copies of J_p, and thus that $J_2 \otimes J_p = J_p \oplus J_p$ as the dimension as a vector space has to equal $2p$.

Suppose that $1 < k < p$. We will show that there is an explicit decomposition

$$J_2 \otimes J_k = \mathbb{K}[t]/(t^2) \otimes \mathbb{K}[t]/(t^k) = \langle (k-1)t \otimes 1 - 1 \otimes t \rangle \oplus \langle 1 \otimes 1 \rangle = J_{k-1} \oplus J_{k+1}, \tag{7.16}$$

where $\langle v \rangle$ is the module generated by the vector v.

It is not hard to see that $t^k \triangleright (1 \otimes 1) = kt \otimes t^{k-1}$ and $t^{k-2} \triangleright ((k-1)t \otimes 1 - 1 \otimes t) = t \otimes t^{k-2} - 1 \otimes t^{k-1}$. This implies that these span copies of J_{k+1} and J_{k-1} respectively, but it does not yet show that these modules are direct summands of $J_2 \otimes J_k$ or that they intersect in zero. If we can show that these copies of J_{k+1} and J_{k-1} are disjoint, then we have found a $2k$-dimensional submodule of $J_2 \otimes J_k$, which implies that we have a decomposition $J_2 \otimes J_k = J_{k-1} \oplus J_{k+1}$. To prove this, we need to prove that the vectors generated by $1 \otimes 1$ and $(k-1)t \otimes 1 - 1 \otimes t$ are linearly independent in every degree. In degree $n \geq 1$ we have $t^n \triangleright (1 \otimes 1) = nt \otimes t^{n-1} + 1 \otimes t^n$ and $t^{n-1} \triangleright ((k-1)t \otimes 1 - 1 \otimes t) = (k-1-n)t \otimes t^{n-1} - 1 \otimes t^n$. These vectors are clearly linearly independent.

We can use this to prove the structure of general tensor products inductively. Indeed, we now have descriptions of $J_k \otimes J_\ell$ for $k = 1, 2$ and ℓ arbitrary, and this follows (7.14). Suppose now that we know $J_n \otimes J_\ell$ for all $n < k$ and all ℓ, we will show that we can deduce the structure of $J_k \otimes J_\ell$. By using the braiding, we see that it is enough to give the structure of $J_k \otimes J_\ell$ when $k < \ell < p$.

We know that $J_2 \otimes J_{k-1} = J_{k-2} \oplus J_k$, which implies that

$$(J_{k-2} \oplus J_k) \otimes J_\ell = J_2 \otimes J_{k-1} \otimes J_\ell. \tag{7.17}$$

We have

$$J_{k-1} \otimes J_\ell = \bigoplus_{i=1}^{\min(k-1,\ell,p-k+1,p-\ell)} J_{|k-1-\ell|+2i-1} \oplus n_{k-1\,\ell} J_p, \tag{7.18}$$

which implies that

$$\begin{aligned} J_2 \otimes J_{k-1} \otimes J_\ell &= \bigoplus_{i=1}^{\min(k-1,\ell,p-k+1,p-\ell)} (J_{|k-1-\ell|+2i-2} \oplus J_{|k-1-\ell|+2i}) \oplus 2n_{k-1\,\ell} J_p \\ &= \bigoplus_{i=1}^{\min(k-1,\ell,p-k+1,p-\ell)} (J_{|k-\ell|+2i-1} \oplus J_{|k-2-\ell|+2i-1}) \oplus 2n_{k-1\,\ell} J_p \\ &= \bigoplus_{i=1}^{\min(k-2,\ell,p-k+2,p-\ell)} J_{|k-2-\ell|+2i-1} \oplus \bigoplus_{i=1}^{\min(k,\ell,p-k,p-\ell)} J_{|k-\ell|+2i-1} \oplus 2n_{k-1\,\ell} J_p \\ &= (J_{k-2} \otimes J_\ell) \oplus (J_k \otimes J_\ell) \end{aligned}. \tag{7.19}$$

Because of the Krull-Schmidt theorem 2.4.4, we know that this implies that $J_k \otimes J_\ell$ does indeed satisfy (7.14).

The indecomposable objects J_k are self-dual as J_k and $J_k{}^*$ are isomorphic as vector spaces and J_k is the unique indecomposable object of vector space dimension k. Alternatively, the antipode S only switches the sign, which also implies that $J_k{}^* \cong J_k$. ■

Table 7.1 shows the tensor product structure of the indecomposable objects in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$ for algebraically closed fields $\mathbb{K}$ of characteristic $p \geq 13$, we have replaced L_k with $\mathbf{k}$ in this table.

⊗	**1**	**2**	**3**	**4**	**5**	**6**	**7**	…
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	…
2	**2**	**1**⊕**3**	**2**⊕**4**	**3**⊕**5**	**4**⊕**6**	**5**⊕**7**	**6**⊕**8**	…
3	**3**	**2**⊕**4**	**1**⊕**3**⊕**5**	**2**⊕**4**⊕**6**	**3**⊕**5**⊕**7**	**4**⊕**6**⊕**8**	**5**⊕**7**⊕**9**	…
4	**4**	**3**⊕**5**	**2**⊕**4**⊕**6**	**1**⊕**3**⊕**5**⊕**7**	**2**⊕**4**⊕**6**⊕**8**	**3**⊕**5**⊕**7**⊕**9**	**4**⊕**6**⊕**8**⊕**10**	…
5	**5**	**4**⊕**6**	**3**⊕**5**⊕**7**	**2**⊕**4**⊕**6**⊕**8**	**1**⊕**3**⊕**5**⊕**7**⊕**9**	**2**⊕**4**⊕**6**⊕**8**⊕**10**	**3**⊕**5**⊕**7**⊕**9**⊕**11**	…
6	**6**	**5**⊕**7**	**4**⊕**6**⊕**8**	**3**⊕**5**⊕**7**⊕**9**	**2**⊕**4**⊕**6**⊕**8**⊕**10**	**1**⊕**3**⊕**5**⊕**7**⊕**9**⊕**11**	**2**⊕**4**⊕**6**⊕**8**⊕**10**⊕**12**	…
7	**7**	**6**⊕**8**	**5**⊕**7**⊕**9**	**4**⊕**6**⊕**8**⊕**10**	**3**⊕**5**⊕**7**⊕**9**⊕**11**	**2**⊕**4**⊕**6**⊕**8**⊕**10**⊕**12**	**1**⊕**3**⊕**5**⊕**7**⊕**9**⊕**11**⊕**13**	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 7.1: Tensor product structure of indecomposable representations of α_p over an algebraically closed field of characteristic $p \geq 13$.

7.2 The Verlinde category Ver_p

7.2.1 The abelian and monoidal structure of Ver_p

The Verlinde category Ver_p is the semisimplification of $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$, in particular the braiding $\overline{\gamma}$ on Ver_p is the image of the braiding γ on ${}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$ (which is just the swap map) under the semisimplification functor ${}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod} \to \mathrm{Ver}_p$. As ${}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$ is a symmetric tensor category, and it has a finite set of (isomorphism classes of) indecomposable objects, we know that Ver_p is a symmetric fusion category. The above results lead to the following proposition.

Proposition 7.2.1. *Let $p > 0$ be prime. The Verlinde category Ver_p is a symmetric fusion category with $p-1$ (isomorphism classes of) simple objects L_k for $k = 1, \ldots, p-1$. L_1 is the monoidal unit for this category, and these simple objects are self-dual, such that $\dim(L_k) = k$, and such that they satisfy the following truncated Clebsch-Gordan rule*

$$L_k \otimes L_\ell = \bigoplus_{i=1}^{\min(k,\ell,p-k,p-\ell)} L_{|k-\ell|+2i-1}. \tag{7.20}$$

Proof. This follows from Lemma 7.1.2 and Lemma 7.1.3 by setting $L_k = \overline{J_k}$, i.e. the image of the indecomposable modules J_k under the semisimplification functor ${}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod} \to \mathrm{Ver}_p$. ■

Tables 7.2, 7.3, 7.4, and 7.5 show the tensor product structure for the Verlinde categories in low characteristics.

$\otimes$	**1**
1	**1**

Table 7.2: Tensor product structure of Ver_2.

$\otimes$	**1**	**2**
1	**1**	**2**
2	**2**	**1**

Table 7.3: Tensor product structure of Ver_3.

$\otimes$	**1**	**2**	**3**	**4**
1	**1**	**2**	**3**	**4**
2	**2**	**1**⊕**3**	**2**⊕**4**	**3**
3	**3**	**2**⊕**4**	**1**⊕**3**	**2**
4	**4**	**3**	**2**	**1**

Table 7.4: Tensor product structure of Ver_5.

$\otimes$	**1**	**2**	**3**	**4**	**5**	**6**
1	**1**	**2**	**3**	**4**	**5**	**6**
2	**2**	**1**⊕**3**	**2**⊕**4**	**3**⊕**5**	**4**⊕**6**	**5**
3	**3**	**2**⊕**4**	**1**⊕**3**⊕**5**	**2**⊕**4**⊕**6**	**3**⊕**5**	**4**
4	**4**	**3**⊕**5**	**2**⊕**4**⊕**6**	**1**⊕**3**⊕**5**	**2**⊕**4**	**3**
5	**5**	**4**⊕**6**	**3**⊕**5**	**2**⊕**4**	**1**⊕**3**	**2**
6	**6**	**5**	**4**	**3**	**2**	**1**

Table 7.5: Tensor product structure of Ver_7.

In an online seminar on tensor categories, Kevin Coulembier gave the following very simple argument that Verlinde categories are not “classical”. Here, we refer to a tensor category as classical when there is a tensor functor to a category of vector spaces. Suppose that there is a tensor functor $F : \mathrm{Ver}_p \to \mathbf{Vect}_{\mathbb{K}}$ for $p \geq 5$. We have an object $L_3 \in \mathrm{Ob}\left(\mathrm{Ver}_p\right)$ for which $L_3 \otimes L_3 = L_1 \oplus L_3$. As F is a tensor functor, we know that $F(L_3) \otimes F(L_3) = F(L_1) \oplus F(L_3)$. As L_1 is the monoidal unit for Ver_p, we conclude that $F(L_1) = \mathbb{K}$. This implies that $\dim(L_3) = \dim(F(L_3))$ satisfies $x^2 = 1 + x$, which would imply that $\dim(L_3)$ is the golden ratio!

7.2.2 Fusion subcategories of Ver_p

As shown in [Ost20, §3], the Verlinde category Ver_p has precisely four fusion subcategories when $p \geq 5$:

1. the category of finite-dimension vector spaces $\mathbf{FinVect}_{\mathbb{K}}$, which is the subcategory generated by the monoidal unit L_1,
2. the category of finite-dimensional super-vector spaces $\mathbf{FinsVect}_{\mathbb{K}}$, which is the subcategory generated by L_1 and L_{p-1}, an equivalence between these categories will follow from Lemma 8.1.1,
3. Ver_p^+, which is defined as the subcategory generated by L_3, this fusion category contains all the odd dimension simple objects,
4. Ver_p.

It is not hard to prove this after noting that a fusion subcategory of Ver_p is equal to Ver_p if and only if it contains a simple object that is of even dimension and not equal to L_{p-1}.

When $p = 2$, these categories all coincide, and when $p = 3$, $\mathbf{sVect}_{\mathbb{K}}$, Ver_3^+ and Ver_3 coincide.

7.3 Alternative constructions for the Verlinde category

7.3.1 Representations of the cyclic group on p elements C_p

Alternatively, the Verlinde category can be constructed as the semisimplification of the finite-dimensional representations of the cyclic group on p elements, C_p, over an algebraically closed field of characteristic $p > 0$.

The Hopf algebra we consider is then the group algebra $\mathbb{K}C_p = \mathbb{K}[x]/(x^p - 1)$, where

1. the multiplication is the standard polynomial multiplication
$$\nabla : \mathbb{K}[x]/(x^p - 1) \otimes \mathbb{K}[x]/(x^p - 1) \to \mathbb{K}[x]/(x^p - 1) : x^m \otimes x^n \mapsto x^{m+n}, \tag{7.21}$$
2. the unit is the standard unit
$$\eta : \mathbb{K} \to \mathbb{K}[x]/(x^p - 1) : \lambda \mapsto \lambda, \tag{7.22}$$
3. the comultiplication is
$$\Delta : \mathbb{K}[x]/(x^p - 1) \to \mathbb{K}[x]/(x^p - 1) \otimes \mathbb{K}[x]/(x^p - 1) : x^n \mapsto x^n \otimes x^n, \tag{7.23}$$
4. the counit is
$$\varepsilon : \mathbb{K}[x]/(x^p - 1) \to \mathbb{K} : 1 \mapsto 1 \text{ and } x^n \mapsto 0 \text{ for } n \geq 1. \tag{7.24}$$

It is clear that the underlying algebra is isomorphic to $\mathbb{K}[t]/(t^p)$. In particular, this implies that the abelian structures of the representation categories of C_p and α_p are equivalent. However, the comultiplication on the Hopf algebras is not the same, which results in different tensor products.

Nevertheless, it turns out that the decomposition of the indecomposable representations follows the same rule as in (7.14) (see [Gre62]). This shows that the semisimplification of this category is equivalent to the semisimplification of the category of representations of α_p.

7.3.2 Tilting modules on $\mathsf{SL}_2(\mathbb{K})$

A final construction of the Verlinde category Ver_p is as the semisimplification of the category of finite-dimensional tilting modules over $\mathsf{SL}_2(\mathbb{K})$ (see, for example, [**Etingof2018**]). We will not discuss this construction, but it is important to mention it, as it is precisely this construction that can be used to construct the higher Verlinde categories Ver_{p^n} (see [BE19; BEO23; Cou21]).

8 Algebras in the Verlinde Category

In this chapter, we discuss certain algebras in the Verlinde category Ver_p. Our approach is inspired by [Kan24; EEK25], focusing primarily on the construction of algebras in Ver_p via the semisimplification functor $S : \mathbf{FinRep}_{\mathbb{K}}(\alpha_p) \to \mathrm{Ver}_p$ applied to algebras which are representations of the linear algebraic group α_p over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$.

Our discussion in the previous Chapter 7 shows that an algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$ is a $\mathbb{K}$-algebra (A, μ), which implies that $\mu : A \otimes_{\mathbb{K}} A \to A$ is a linear map, such that

1. A is a $\mathbb{K}[t]/(t^p)$-module,
2. $\mu : A \otimes_{\mathbb{K}} A \to A$ is a homomorphism of modules, i.e. $\mu(t \triangleright (x \otimes y)) = t \triangleright \mu(x \otimes y)$, which becomes the following in standard notation

$$(tx)y + x(ty) = t(xy). \tag{8.1}$$

We conclude that a $\mathbb{K}$-algebra is an algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$ if and only if it is equipped with a nilpotent derivation of order $\leq p$.

Remark 8.0.1. Note that a nilpotent derivation ∂ of order $\leq p$ defines an automorphism of order p by setting $\sigma := \exp(\partial) = \sum_{n=0}^{p-1} \frac{1}{n!}\partial^n$. Indeed, this morphism has the inverse $\exp(-\partial) = \sum_{m=0}^{p-1} \frac{(-1)^m}{m!}\partial^m$, and

$$\begin{aligned}
\sigma(xy) &= \sum_{n=0}^{p-1} \frac{1}{n!}\partial^n(xy) \\
&= \sum_{n=0}^{p-1} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (\partial^k x)(\partial^{n-k} y) \\
&= \sum_{n=0}^{p-1} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} (\partial^k x)(\partial^{n-k} y) \\
&= \sum_{k=0}^{p-1} \sum_{m=0}^{p-1} \frac{1}{k!m!} (\partial^k x)(\partial^m y) \\
&= \sigma(x)\sigma(y)
\end{aligned}. \tag{8.2}$$

Applying this procedure to Lie or Jordan algebras (A, μ) equipped with nilpotent derivation or automorphisms of order p, we can construct Lie and Jordan superalgebras $(\overline{A}_s, \overline{\mu}_s)$ by taking the component of the semisimplified algebra $(\overline{A}, \overline{\mu}) = (S(A), S(\mu))$ in the fusion subcategory $\mathbf{sVect}_{\mathbb{K}} \subseteq \mathrm{Ver}_p$. This is the construction used in [Kan24; EEK25].

8.1 Simple objects in Ver_p

In this section we plan to study algebras on simple objects in Ver_p. Note that algebras in abelian categories enriched over $\mathbf{FinVect}_{\mathbb{K}}$, where $\mathbb{K}$ is an algebraically closed field, always have trivial automorphism groups by Schur's lemma 2.4.10.

Before turning to the construction of these algebras, we first examine the braiding on Ver_p induced by the braiding (swap map) on $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$. This is motivated by the fact that braidings are one of the main tools that allow us to extract structural information about algebras in symmetric tensor categories.

8.1.1 The braiding on simple objects

The following lemma can be seen as a generalisation of [Kan24, Proposition 3.2.1], although the proof is quite different.

Lemma 8.1.1. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$, and let Ver_p be the Verlinde category over $\mathbb{K}$. Let L_k be the simple object of dimension k in Ver_p, and let $\overline{\gamma}$ be the braiding on Ver_p. There exists a decomposition into simple objects*

$$L_k \otimes L_k = \bigoplus_{i=1}^{\min(k,p-k)} L_{2i-1} \tag{8.3}$$

such that $\overline{\gamma}_{(L_k,L_k)} \circ \mathrm{inc}_{L_{2i-1}} = (-1)^{k-i}\, \mathrm{inc}_{L_{2i-1}}$ and thus such that

$$\overline{\gamma}_{(L_k,L_k)} = \sum_{i=1}^{\min(k,p-k)} (-1)^{k-i}\, \mathrm{inc}_{L_{2i-1}} \circ \mathrm{proj}_{L_{2i-1}}. \tag{8.4}$$

Proof. We will prove a stronger statement in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$. Let γ be the braiding on $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$, i.e. the swap map.

Recall from Lemma 7.1.2 and Lemma 7.1.3 that the indecomposables are uniquely indexed by their dimension and given by $J_m = \mathbb{K}[t]/(t^m)$, and that there is a direct sum decomposition

$$J_m \otimes J_n = \bigoplus_{i=1}^{\min(m,n,p-m,p-n)} J_{2i-1} \oplus n_{mn} J_p. \tag{8.5}$$

We will prove that for any indecomposable module $J_k = \mathbb{K}[t]/(t^k)$ over $R = J_p = \mathbb{K}[t]/(t^p)$, there exists a decomposition into indecomposable modules

$$M = J_k \otimes J_k = M_0 \oplus \cdots \oplus M_{k-1} \tag{8.6}$$

such that

$$M_i \cong J_{2i+1} \text{ for } i < \min(k, p-k), \text{ and } M_i \cong J_p \text{ for } i \geq \min(k, p-k), \tag{8.7}$$

and such that

$$\gamma_{(J_k,J_k)} = \sum_{i=0}^{k-1} (-1)^{k-1-i}\, \mathrm{inc}_{M_i} \circ \mathrm{proj}_{M_i}. \tag{8.8}$$

The proof of this statement will be split into four different steps.

Step 1: We will first discuss the structure of the kernel of the action of t on M.

We have

$$\begin{aligned} t \triangleright \left(\sum_{i,j=0}^{k-1} a_{ij} t^i \otimes t^j \right) &= \sum_{i,j=0}^{k-1} a_{ij} \left(t^{i+1} \otimes t^j + t^i \otimes t^{j+1} \right) \\ &= \sum_{i=1,j=0}^{k-1} a_{i-1\,j} t^i \otimes t^j + \sum_{i=0,j=1}^{k-1} a_{i\,j-1} t^i \otimes t^j \\ &= \sum_{i=0}^{k-2} (a_{i0} t^{i+1} \otimes 1 + a_{0i} 1 \otimes t^{i+1}) + \sum_{i,j=1}^{k-1} (a_{i-1\,j} + a_{i\,j-1}) t^i \otimes t^j. \end{aligned} \tag{8.9}$$

This implies that the kernel of the action of t consists of vectors $\sum_{i,j=0}^{k-1} a_{ij} t^i \otimes t^j$ such that $a_{i-1\,j} = -a_{i\,j-1}$ for all $i, j \geq 1$, and that $a_{i0}, a_{0i} = 0$ for all $i = 0, \ldots, k-2$.

This implies that if we express this vector as a matrix (a_{ij}), the kernel looks something like an upscaled version of

$$\operatorname{Ker}(t \triangleright -) = \left\{ \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & b \\ 0 & a & -b & c \\ -a & b & -c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{K} \right\}. \tag{8.10}$$

The kernel is thus spanned by vectors corresponding to the anti-diagonals in the above matrix

$$z_n := t^n \otimes t^{k-1} - t^{n+1} \otimes t^{k-2} + \cdots + (-1)^{k-1-n} t^{k-1} \otimes t^n \text{ for } n = 0, \ldots, k-1. \tag{8.11}$$

Step 2: We will now prove that M admits a decomposition into indecomposable modules that are generated by a symmetric or skew-symmetric homogeneous vector.

The ring $R = J_p = \mathbb{K}[t]/(t^p)$ and the indecomposable R-modules $J_m = \mathbb{K}[t]/(t^m)$ are graded by the degree of polynomials, i.e. $J_m = J_m^0 \oplus J_m^1 \oplus \cdots \oplus J_m^{p-1}$ with J_m^ℓ the vector space spanned by t^ℓ. Similarly, the R-module M has a grading $M = M_0 \oplus M_1 \oplus \cdots \oplus M_{2k-2}$, by using the degree $\deg(t^i \otimes t^j) = i + j$. t maps M_i to M_{i+1}, which shows that M equipped with the above grading is indeed a graded module over the graded ring R. The action of t on M then gives a morphism in the category of graded modules (from M to the shift of M).

More generally, any R-module is graded by using the decomposition into indecomposable modules (which are all graded). The indecomposable objects in the category of graded R-modules are then clearly the same as the indecomposable objects in the category of R-modules.

As the category of graded modules is still Krull-Schmidt, we obtain a decomposition into indecomposable graded R-modules

$$M = M_1 \oplus \cdots \oplus M_n. \tag{8.12}$$

As we know that $M = S^2 M \oplus \wedge^2 M$ (Corollary 4.5.6) we may assume that the M_i are either contained in $S^2 M$ or $\wedge^2 M$, which implies that their generators are symmetric or skew-symmetric.

Now, $M_i \cong J_{m_i}$ for some $m_i \in \{1, \ldots, p\}$. M_i is graded by the degree as a graded submodule of the graded module M. Let x_i be an arbitrary vector in the lowest degree summand of M_i. As x_i is of lowest degree in M_i, we know that any generator of M_i (and there exists one as $M_i \cong J_{m_i}$) has to be a scalar multiple of x_i. In particular, x_i is a generator.

Step 3: We combine the first two steps to figure out which indecomposables are symmetric and which ones are skew-symmetric.

We know that $t^{m_i - 1} x_i \in \operatorname{Ker}(t) \setminus \{0\}$, which implies without loss of generality that $t^{m_i - 1} x_i = z_j$ for some $j \in \{0, \ldots, k-1\}$ as these are the only homogeneous vectors in the kernel.

This implies that the direct summands are labelled by the z_i. In particular: there are k indecomposables in the decomposition. From now on M_i will be the indecomposable module corresponding to z_i.

It is now clear that the symmetry or skew-symmetry of z_i determines the symmetry or skew-symmetry of M_i. The M_i such that z_i is symmetric satisfy $\gamma_{(J_k, J_k)} \circ \operatorname{inc}_{M_i} = \operatorname{inc}_{M_i}$, and the M_i such that z_i is skew-symmetric satisfy $\gamma_{(J_k, J_k)} \circ \operatorname{inc}_{M_i} = -\operatorname{inc}_{M_i}$.

This implies that $M_{k-1}, M_{k-3}, \ldots$ are symmetric, and that $M_{k-2}, M_{k-4}, \ldots$ are skew-symmetric.

We thus find

$$\gamma_{(J_k, J_k)} = \sum_{i=0}^{k-1} (-1)^{k-1-i} \operatorname{inc}_{M_i} \circ \operatorname{proj}_{M_i}. \tag{8.13}$$

Step 4: We will figure out which indecomposable J_m corresponds to which M_i.

Suppose first that $k \leq p - k$. We then know that

$$J_k \otimes J_k = J_1 \oplus J_3 \oplus \cdots \oplus J_{2k-1}. \tag{8.14}$$

The homogeneous vector z_n in the kernel is a vector of degree $n + k - 1$. In particular, the only such vector with a degree of order $\geq 2k - 2$ is z_{k-1}. This implies that $M_{k-1} \cong J_{2k-1}$ as z_{k-1} is the only homogeneous vector in the kernel that could possibly be obtained through $t^{2k-2} \triangleright -$.

More generally, the copy of $J_{2(k-i)-1}$ has to be generated by a vector of degree i or smaller (for any other vector v we find $t^{2(k-i)-2} \triangleright v = 0$). This implies that the degree of the corresponding z_j has to be in between $2k - i - 2$ and $2k - 2$, i.e. it has to be one of $z_{k-i-1}, z_{k-i-2}, \ldots, z_{k-1}$. Induction shows that $z_{k-1}, z_{k-2}, \ldots, z_{k-i}$ (of degree $2k-2, 2k-3, \ldots, 2k-i-1$) already correspond to a copy of $J_{2(k-j)-1}$ with j strictly smaller than i. This shows that $J_{2(k-i)-1}$ has to correspond to z_{k-i-1}, i.e. M_{k-i-1}.

We conclude

$$M_i \cong J_{2i+1}. \tag{8.15}$$

Suppose now that $k > p - k$. We then have

$$J_k \otimes J_k = J_1 \oplus J_3 \oplus \cdots \oplus J_{2(p-k)-1} \oplus (p - 2k) J_p. \tag{8.16}$$

As before, a generator of $M_i \cong J_p$ has to have degree $\leq i + k - p$. This is a negative number for $i < p - k$, which implies that the indecomposable modules $M_0, M_1, \ldots, M_{p-k-1}$ correspond to $J_1, J_3, \ldots, J_{2(p-k)-1}$ (not necessarily in that order yet). This gives an exhaustive list for these modules as there are $p - k$ modules in each list, and we can thus conclude that the indecomposable modules $M_{p-k}, M_{p-k+1}, \ldots, M_{k-1}$ are all isomorphic to J_p. Continuing in the same way from M_{p-k-1} downward, we conclude that

$$M_i \cong J_{2i+1} \text{ for } i < p - k, \text{ and } M_i \cong J_p \text{ for } i \geq p - k. \tag{8.17}$$

■

Remark 8.1.2. The above lemma shows that $L_k \otimes L_k$ and $L_{p-k} \otimes L_{p-k}$ are isomorphic objects but have braidings that only agree up to a sign. In particular, applying this to $L_1 \otimes L_1 = L_1$ (which has trivial braiding through Lemma 3.6.5) and $L_{p-1} \otimes L_{p-1} = L_1$, this shows that the fusion subcategory spanned by L_1 and L_{p-1} is symmetric tensor equivalent to $\mathbf{sVect}_{\mathbb{K}}$.

Moreover, this shows that $S^n L_k = \wedge^n L_{p-k}$ and $\wedge^n L_k = S^n L_{p-k}$ for all even n. $L_k \otimes L_k \cong L_{p-k} \otimes L_{p-k}$ shows that $\otimes^n L_k \cong \otimes^n L_{p-k}$ whenever n is even. Lemma 8.1.1 shows that $\overline{\gamma}_{(L_k, L_k)} = -\overline{\gamma}_{(L_{p-k}, L_{p-k})}$, which shows that the action of an element $\sigma \in S_n$ is trivial on $\otimes^n L_k$ if and only if the action of $\operatorname{sgn}(\sigma)\sigma$ is trivial on $\otimes^n L_{p-k}$. This then implies that $\otimes^n L_k = \otimes^n L_{p-k} \xrightarrow{f} X$ is a coequaliser of the morphisms $\sigma : \otimes^n L_k \to \otimes^n L_k$ if and only if it is a coequaliser of the morphisms $\operatorname{sgn}(\sigma)\sigma : \otimes^n L_{p-k} \to \otimes^n L_{p-k}$. We conclude that $S^n L_k = \wedge^n L_{p-k}$.

Note that we do indeed find

$$\begin{aligned} \dim(\wedge^n L_{p-k}) &= \binom{p-k}{n} = \frac{(p-k)(p-k-1)\cdots(p-k-n+1)}{n!} \\ &= (-1)^n \frac{k(k+1)\cdots(k+n-1)}{n!} = \binom{k+n-1}{n} = \dim(S^n L_k) \end{aligned}. \tag{8.18}$$

The above lemma also allows us to explicitly determine the symmetric and exterior square of a simple object.

Proposition 8.1.3. *Let $p > 2$ be prime and let L_k be the simple object of dimension k in Ver_p. We have*

$$S^2 L_k = \bigoplus_{i=1}^{\frac{\min(k,p-k)-1}{2}} L_{4i-3} \text{ and } \wedge^2 L_k = \bigoplus_{i=1}^{\frac{\min(k,p-k)}{2}} L_{4i-1} \text{ if } k \text{ is odd}, \tag{8.19}$$

$$S^2 L_k = \bigoplus_{i=1}^{\frac{\min(k,p-k)}{2}} L_{4i-1} \text{ and } \wedge^2 L_k = \bigoplus_{i=1}^{\frac{\min(k,p-k)-1}{2}} L_{4i-3} \text{ if } k \text{ is even}. \tag{8.20}$$

Proof. S^2L_k is the coequaliser of $\mathrm{id}_{L_k\otimes L_k}$ and $\overline{\gamma}_{(L_k,L_k)}$, and $\wedge^2 L_k$ is the coequaliser of $\mathrm{id}_{L_k\otimes L_k}$ and $-\overline{\gamma}_{(L_k,L_k)}$. In particular, any direct summand of S^2L_k should be such that $\overline{\gamma}_{(L_k,L_k)}\circ \mathrm{inc} = \mathrm{inc}$, and any direct summand of $\wedge^2 L_k$ should be such that $\overline{\gamma}_{(L_k,L_k)}\circ \mathrm{inc} = -\,\mathrm{inc}$. Lemma 8.1.1 now shows that the above decompositions hold. ■

8.1.2 Algebras on simple objects

To understand the behaviour of algebras in any symmetric fusion category, a natural first step is to study the algebras defined on simple objects. Such algebras always have trivial automorphism groups by Schur's lemma 2.4.10. In $\mathbf{FinVect}_{\mathbb{K}}$, these algebras are rather trivial, since the only simple object is $\mathbb{K}$. In contrast, the category Ver_p contains multiple simple objects, which opens the possibility of constructing non-trivial algebra structures on them.

On which simple objects do non-zero algebra structures exist?

Lemma 8.1.4. *Let $p>2$ be prime. The only simple objects that admit a non-zero algebra structure are those L_k for which k is odd and $3k\leq 2p-1$. This algebra structure is then unique up to isomorphism.*

If k is odd, and $3k\leq 2p-1$, then

1. *if $\frac{k-1}{2}$ is even, then the unique algebra structure on L_k is commutative, this means that the algebras on $L_1, L_5, L_9, \ldots$ are commutative,*
2. *if $\frac{k-1}{2}$ is odd, then then unique algebra structure on L_k is anti-commutative, this means that the algebras on $L_3, L_7, L_{11}, \ldots$ are anti-commutative.*

Proof. Let $\mu : L_k\otimes L_k \to L_k$ be a morphism. For any decomposition $L_k\otimes L_k = X_1\oplus\cdots\oplus X_n$ into simple objects, we have $\mu = \mu\circ(\mathrm{inc}_{X_1}\circ\mathrm{proj}_{X_1}+\cdots+\mathrm{inc}_{X_n}\circ\mathrm{proj}_{X_n})$. Schur's lemma 2.4.9 implies that $\mu\circ\mathrm{inc}_{X_i}$ is non-zero if and only if $X_i\cong L_k$, and $\mu\circ\mathrm{inc}_{X_i}$ is then just a scalar through Schur's lemma over algebraically closed fields 2.4.10.

Recall

$$L_k\otimes L_k = \bigoplus_{i=1}^{\min(k,p-k)} L_{2i-1}. \tag{8.21}$$

There is a copy of L_k in any decomposition of $L_k\otimes L_k$ into simple objects if and only if k is odd and $\frac{k+1}{2}\leq p-k$ (equivalently, $3k\leq 2p-1$), and there is then exactly one copy. This implies that $\mu=\mu\circ\mathrm{inc}_{L_k}\circ\mathrm{proj}_{L_k}$, and is hence the composition of an isomorphism and proj_{L_k}. This proves the first statement.

Suppose that k is odd and $3k\leq 2p-1$, and that we are provided with the decomposition from Lemma 8.1.1. Let $\mu=\mathrm{proj}_{L_k}$ be the unique morphism up to isomorphism. Lemma 8.1.1 shows that

$$\mu\circ\overline{\gamma}_{(L_k,L_k)} = (-1)^{\frac{k-1}{2}}\mu, \tag{8.22}$$

which proves the result. ■

Remark 8.1.5. Note that Schur's lemma 2.4.9 implies that any algebra structure on a non-zero simple object not isomorphic to the monoidal unit cannot be unital.

Example 57. We perform some explicit computations of algebras on simple objects in Ver_p. Since this is currently the only meaningful way to gain insight into these algebras, we carry out these computations in the parent category $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$.

As L_1 is the monoidal unit, we know that the unique non-zero algebra structure on this object is $(L_1, \rho_{L_1} = \lambda_{L_1})$.

Lemma 8.1.4 shows that there is a unique non-zero algebra structure on L_3 if and only if $9 \leq 2p-1$, i.e. if $p \geq 5$. Lemma 8.1.1 shows that we have the following decomposition in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$ for any algebraically closed field $\mathbb{K}$ of characteristic $p \geq 5$

$$J_3 \otimes J_3 = J_1 \oplus J_3 \oplus J_5 = \langle 1 \otimes t^2 - t \otimes t + t^2 \otimes 1 \rangle \oplus \langle 1 \otimes t - t \otimes 1 \rangle \oplus \langle 1 \otimes 1 \rangle. \tag{8.23}$$

The unique non-zero algebra structure on L_3 lifts to proj_{J_3}, which implies that

$$\left(a + bt + ct^2\right) \cdot \left(\overline{a} + \overline{b}t + \overline{c}t^2\right) = \frac{1}{2}\left((a\overline{b} - \overline{a}b) + (a\overline{c} - c\overline{a})t + (b\overline{c} - \overline{b}c)t^2\right). \tag{8.24}$$

Any linear equation satisfied by this algebra will also be satisfied by the semisimplified algebra on L_3. In particular, as we will see below, this is a Lie algebra. Note that this algebra is not associative.

Similarly, one obtains decompositions and algebras on $J_5, J_7, \ldots$

The structure of algebras on simple objects

To prove that the unique algebras on simple objects have a certain structure, we first introduce the notion of external dimension (which essentially coincides with the usual vector space dimension), and establish a few preliminary results using this concept.

Definition 8.1.6 (External and internal dimension of objects). Let $\mathcal{C}$ be a symmetric tensor category over a field of characteristic $p > 0$ in which every object admits a decomposition into indecomposable objects. Proposition 4.5.3 shows that $\dim(A) \in \mathbb{F}_p$ for any object $A \in \mathrm{Ob}(\mathcal{C})$.

Let $X \in \mathrm{Ob}(\mathcal{C})$ be an indecomposable object. We define the *external dimension* of X, denoted $\mathrm{extdim}(X)$, to be the unique integer in $\{0, 1, \ldots, p-1\}$ corresponding to $\dim(X) \in \mathbb{F}_p$.

Now let $A \in \mathrm{Ob}(\mathcal{C})$ be an arbitrary object with a decomposition $A \cong X_1 \oplus \cdots \oplus X_n$ into indecomposable summands. We define

$$\mathrm{extdim}(A) = \sum_{k=1}^{n} \mathrm{extdim}(X_k), \tag{8.25}$$

which is well-defined by the Krull-Schmidt theorem 2.4.4.

Note that this external dimension satisfies $\dim(A) \simeq \mathrm{extdim}(A) \mod p$, where $\dim(A)$ denotes the "internal" dimension as introduced in Chapter 3.

Lemma 8.1.7. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$, and let $F : \mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod} \to \mathbf{FinVect}_{\mathbb{K}}$ be the forgetful functor. For any $A \in \mathrm{Ob}\left(\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)\right)$ we have*

$$\mathrm{extdim}_{\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)}(A) = \mathrm{extdim}_{\mathbf{FinVect}_{\mathbb{K}}}(F(A)). \tag{8.26}$$

Proof. As F is braided monoidal, we find $F(\dim_{\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)}(A)) = \dim_{\mathbf{FinVect}_{\mathbb{K}}}(F(A))$. As F is also faithful, we then find $\dim_{\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)}(A) = \dim_{\mathbf{FinVect}_{\mathbb{K}}}(F(A))$.

This then implies that $\mathrm{extdim}_{\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)}(A) = \mathrm{extdim}_{\mathbf{FinVect}_{\mathbb{K}}}(F(A))$ for any $A \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{extdim}_{\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)}(A) < p$, in particular for indecomposable objects. Indeed, the indecomposable $\mathbb{K}[\alpha_p]$-modules $\mathbb{K}[t]/(t^k)$ have external and internal dimension k as $\mathbb{K}$-vector spaces.

Now, suppose that $A = X_1 \oplus \cdots \oplus X_n$ with X_k indecomposable. We then find

$$\mathrm{extdim}(A) = \sum_{k=1}^{n} \mathrm{extdim}(X_k) = \sum_{k=1}^{n} \mathrm{extdim}(F(X_k)) = \mathrm{extdim}(F(A)). \tag{8.27}$$

■

Proposition 8.1.8. *Let p be prime. For any $A \in \mathrm{Ob}\left(\mathrm{Ver}_p\right)$ with $\mathrm{extdim}(A) < p$, we have*

$$\wedge^{\mathrm{extdim}(A)} A = \mathbb{1}. \tag{8.28}$$

In particular, for the simple objects L_k in Ver_p, $\wedge^k L_k = L_1 = \mathbb{1}$.

Proof. Let $X \in \mathrm{Ob}\left(\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)\right)$ be such that $S(X) = A$ and $\mathrm{extdim}(X) = \mathrm{extdim}(A)$, where $S : \mathbf{FinRep}_{\mathbb{K}}(\alpha_p) \to \mathrm{Ver}_p$ is the semisimplification functor.

Let $n \in \mathbb{N}$, and let $a_n : X^{\otimes n} \to X^{\otimes n}$ be the skew-symmetriser introduced in Proposition 4.5.5. As the semisimplification functor $S : \mathbf{FinRep}_{\mathbb{K}}(\alpha_p) \to \mathrm{Ver}_p$ is braided monoidal and additive, we know that $\overline{a}_n = S(a_n)$ is the skew-symmetriser on $A^{\otimes n}$.

$\mathrm{coim}(a_n)$ is a split epimorphism (Lemma 4.5.7), which implies that $\mathrm{coim}(S(a_n)) = S(\mathrm{coim}(a_n))$. Proposition 4.5.5 then shows that $S(\wedge^n X) = \wedge^n A$.

Setting $n = \mathrm{extdim}(A) = \mathrm{extdim}(X)$, we know that $\wedge^{\mathrm{extdim}(X)} X = \mathbb{1}$, hence that $\wedge^{\mathrm{extdim}(A)} A = \mathbb{1}$. ■

Proposition 8.1.8 allows us to prove a result about the structure of the algebras on simple objects.

Definition 8.1.9 (Generalised Lie algebras). Let $\mathcal{C}$ be a symmetric tensor category. We call an algebra (A, μ) in $\mathcal{C}$ a *generalised n-Lie algebra* (with $n+2$ smaller than the characteristic of the field if it is non-zero) if

1. (A, μ) is anti-commutative,
2. μ is such that for any homogeneous polynomial $p(\mu) : A^{\otimes(n+2)} \to A^{\otimes(n+2)}$ of degree $n+1$ in μ, constructed using compositions and tensor products of μ and id_A[1], we have

$$p(\mu) \circ a_{n+2} = 0, \text{ where } a_{n+2} : A^{\otimes(n+2)} \to A^{\otimes(n+2)} \text{ is the skew-symmetriser.} \tag{8.29}$$

We call an algebra (A, μ) a *generalised n-anti-Lie algebra* if

1. (A, μ) is commutative,
2. μ is such that for any homogeneous polynomial $p(\mu) : A^{\otimes(n+2)} \to A^{\otimes(n+2)}$ of degree $n+1$ in μ, constructed using compositions and tensor products of μ and id_A, we have

$$p(\mu) \circ s_{n+2} = 0, \text{ where } s_{n+2} : A^{\otimes(n+2)} \to A^{\otimes(n+2)} \text{ is the symmetriser.} \tag{8.30}$$

Remark 8.1.10. Let $\mathbb{K}$ be a field that is not of characteristic two. In S_3 we have $(12)(23) = (123), (13)(23) = (132) = (123)^2, (23)(23) = 1$. For an anti-commutative algebra (A, μ), we have $\mu \circ (\mathrm{id}_A \otimes \mu) \circ \tau = -\mu \circ (\mathrm{id}_A \otimes \mu)$ where τ represents (23). This then implies that $\mu \circ (\mathrm{id}_A \otimes \mu) \circ a_3 = 2\mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_{A\otimes A\otimes A} + \sigma + \sigma^2)$ where σ represents (123) or (132).

Also, $\mu \circ (\mathrm{id}_A \otimes \mu) = -\mu \circ \gamma_{(A,A)} \circ (\mathrm{id}_A \otimes \mu) = -\mu \circ (\mu \otimes \mathrm{id}_A) \circ \gamma_{(A,A\otimes A)}$, which shows that the Jacobi identities on $\mu \circ (\mathrm{id}_A \otimes \mu)$ and $\mu \circ (\mu \otimes \mathrm{id}_A)$ are equivalent.

We conclude that a generalised 1-Lie algebra is simply a Lie algebra.

Corollary 8.1.11. *Let $p > 2$ be prime and let L_k be a simple object of odd dimension k in Ver_p that admits an algebra structure.*

1. *If $\frac{k-1}{2}$ is odd, then L_k equipped with the unique non-zero algebra structure is a generalised $(k-2)$-Lie algebra.*
2. *If $\frac{k-1}{2}$ is even, then L_k equipped with the unique non-zero algebra structure is a generalised anti-$(p - k - 2)$-Lie algebra.*

[1] In degree one we have μ, in degree two we have linear combinations of $\mu \circ (\mathrm{id}_A \otimes \mu)$ and $\mu \circ (\mu \otimes \mathrm{id}_A)$, and so forth.

Proof. Suppose first that $\frac{k-1}{2}$ is odd. Lemma 8.1.4 shows that the algebra structure on L_k is anti-commutative. Let $f : L_k^{\otimes k} \to L_k$ be any morphism. We find $f \circ a_k = 0$ as it factors through $\operatorname{Coim}(a_k) = \wedge^k L_k = \mathbb{1}$ (Proposition 8.1.8). In particular, this is true when setting $f = p(\mu)$ as in Definition 8.1.9.

When $\frac{k-1}{2}$ is even, we find $S^{p-k} L_k \cong \wedge^{p-k} L_{p-k} = \mathbb{1}$ (Remark 8.1.2), and the result follows with a similar proof. ∎

Remark 8.1.12. More generally, any anti-commutative algebra (A, μ) in Ver_p with $\operatorname{extdim}(A) < p$ that does not contain $\mathbb{1}$ is a generalised $(\operatorname{extdim}(A) - 2)$-Lie algebra.

We can now list the algebras on simple objects in some small characteristics.

1. In $\mathrm{Ver}_2 \simeq \mathbf{FinVect}_{\mathbb{K}}$ and $\mathrm{Ver}_3 \simeq \mathbf{FinsVect}_{\mathbb{K}}$ there is only the trivial monoidal unit algebra on L_1.
2. Ver_5 and Ver_7 have two non-zero algebras on simple objects: the trivial one on L_1, and a Lie algebra on L_3.
3. Ver_{11} has four non-zero algebras: the trivial one on L_1, a Lie algebra on L_3, a generalised 3-anti-Lie algebra on L_5, and a generalised 5-Lie algebra on L_7.

We have computational evidence, obtained using `SageMath`, that the unique algebra on L_7 satisfies the following generalised Jacobi identity

$$\big(\mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_{A\otimes A} \otimes \mu) + \mu \circ (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_A \otimes \mu \otimes \mathrm{id}_A) + \mu \circ (\mu \otimes \mu)\big) \circ \text{cyclic permutations} = 0. \tag{8.31}$$

This is an equation in 4 variables, and not in 7 variables like in the above. In Ver_{11}, it might be possible to explain this through its interaction with L_4 (as $4 = 11 - 7$, see Remark 8.1.2). However, even if this explanation holds in this specific case, this does not give an explanation in higher characteristics. It is thus clear that our discussion is far from complete.

Algebras in Ver_p obtained from algebras on simple objects

Let $A \in \mathrm{Ob}\left(\mathrm{Ver}_p\right)$ be an arbitrary object. This object has a decomposition into simple objects $A = n_1 L_1 \oplus \cdots \oplus n_{p-1} L_{p-1}$, which is unique up to isomorphism through the Krull-Schmidt theorem 2.4.4.

The unique algebra structure on simple objects allows us to define special algebra structures on A. Given a subset $\mathcal{S}$ of the indices k, i such that there is a copy $L_k^i \cong L_k$ in the direct sum decomposition, we define $\mu_{\mathcal{S}} : A \otimes A \to A$ by setting

$$\mathrm{proj}^i_{L_k} \circ \mu_{\mathcal{S}} \circ (\mathrm{inc}^i_{L_k} \otimes \mathrm{inc}^i_{L_k}) = \mu_{L_k} \text{ for all } (k,i) \in \mathcal{S}, \text{ and setting all other components to zero.} \tag{8.32}$$

This implies that

$$\mu_{\mathcal{S}} = \sum_{(k,i)\in\mathcal{S}} \mathrm{inc}^i_{L_k} \circ \mu_{L_k} \circ (\mathrm{proj}^i_{L_k} \otimes \mathrm{proj}^i_{L_k}) \tag{8.33}$$

These algebras can have non-trivial automorphism groups, we can permute different copies corresponding to the same simple object.

8.2 On the semisimplification of algebras

8.2.1 A conjecture on Lie algebras in Ver_p

Kannan's construction of exceptional Lie superalgebras begs the question how powerful the semisimplification process on Lie algebras is. More specifically, the following question sparks attention: "Can every Lie algebra in Ver_p be obtained as the semisimplification of a Lie algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$?"

The converse is definitely true: any algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$ satisfying a set of linear axioms (i.e. axioms which can also be expressed in tensor categories) gets mapped to an algebra satisfying the same axioms in

Ver_p. In particular, this implies that Lie algebras get mapped to Lie algebras. Nonetheless, there could be Lie algebras in Ver_p for which any *parent* (i.e. an algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$ that gets mapped to this algebra in Ver_p after semisimplification) is not a Lie algebra.

We conjecture the following.

Conjecture 8.2.1. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 2$ and let Ver_p be the semisimplification of $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$, i.e. the Verlinde category. There exist Lie algebras in Ver_p that cannot be constructed as the semisimplification of a Lie algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$, i.e. a Lie algebra equipped with a nilpotent derivation of order $\leq p$.*

At first glance, it may seem obvious that this conjecture must be true: the Verlinde category is significantly richer than the classical categories $\mathbf{Vect}_{\mathbb{K}}$ and $\mathbf{sVect}_{\mathbb{K}}$, so how could every Lie algebra in such a richer setting arise from a classical Lie algebra?

However, the question is more subtle than it appears. One must take into account the fact that we can freely add any number of copies of J_p to any parent (or, more generally, any negligible morphism). More precisely, consider an algebra $(\overline{A}, \overline{\mu})$ in Ver_p, and let (A, μ) be any parent of $(\overline{A}, \overline{\mu})$. Then *any* algebra (A', μ') with $A' = A \oplus nJ_p$, and such that $\mu = \mathrm{proj}_A \circ \mu' \circ (\mathrm{inc}_A \otimes \mathrm{inc}_A)$, also maps to $(\overline{A}, \overline{\mu})$ under the semisimplification functor. This allows for a lot of freedom!

Remark 8.2.2. Adding multiple copies of J_p to a parent was a strategy the author attempted in an effort to disprove Conjecture 8.2.1. However, an important verification (that an algebra structure was well-defined) was overlooked, rendering the argument invalid.

A second attempt, this time aiming to prove Conjecture 8.2.1, focused on Lie algebras satisfying the celebrated Poincaré–Birkhoff–Witt (PBW) theorem. It is well known (see [Eti18; Ven23]) that not every Lie algebra in Ver_p satisfies the PBW theorem, whereas every Lie algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$ does (for $p > 2$). So, once could hope that the semisimplification functor preserves the PBW property. However, this approach quickly encounters difficulties: the semisimplification functor is not exact, which implies that the universal enveloping algebra of a Lie algebra in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$ is not guaranteed to map to the universal enveloping algebra of the corresponding Lie algebra in Ver_p.

Perhaps the most compelling evidence in support of Conjecture 8.2.1 is the following theorem, which shows that affine algebraic groups in Ver_p can be described as a pair consisting of a classical affine (linear) algebraic group and a Lie algebra in Ver_p satisfying the PBW theorem.

Theorem 8.2.3 ([Ven23, Theorem 1.2]). *The category of affine algebraic groups in $\mathrm{Ver}_p^{\mathrm{ind}}$ is equivalent to the category of Harish-Chandra pairs in Ver_p. Harish-Chandra pairs are pairs $(G_0, \mathfrak{g})$ of an affine algebraic group G_0 over $\mathbb{K}$ and a Lie algebra $\mathfrak{g}$ in Ver_p such that the induced Lie algebra induced on the copies of the monoidal unit corresponds to the Lie algebra of G_0, i.e. $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$.*

The Harish-Chandra pair corresponding to an affine algebraic group G in $\mathrm{Ver}_p^{\mathrm{ind}}$ is $(G_0, \mathrm{Lie}(G))$, where G_0 is the linear algebraic group obtained by restricting to the fusion subcategory generated by the monoidal unit.

If Conjecture 8.2.1 holds true, then it could potentially be straightforward to give a negative answer to [CEO24a, Question 4.6]. This question asks whether all *invariantless Lie algebras* in Ver_p (i.e. Lie algebras (A, μ) such that $\mathrm{Hom}_{\mathrm{Ver}_p}(\mathbb{1}, A) = 0$, or equivalently such that A does not contain a direct summand isomorphic to $\mathbb{1}$) can be constructed as the semisimplification of certain Lie algebras equipped with particular derivations.

8.2.2 A weaker version of our conjecture on Lie algebras

Our interest lies in the structure of more general algebras in Ver_p, not just Lie algebras. This motivates the following weaker version of Conjecture 8.2.1. The author would be very surprised if this conjecture turned out to be false.

Conjecture 8.2.4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 2$ and let Ver_p be the semisimplification of $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p) = {}_{\mathbb{K}[\alpha_p]}\mathbf{FinMod}$, i.e. the Verlinde category. There exists a type of algebra (defined in the general setting of symmetric tensor categories) for which there is an instance in Ver_p that cannot be obtained as the semisimplification of an algebra of the same type in $\mathbf{FinRep}_{\mathbb{K}}(\alpha_p)$.*

If this conjecture holds true, then it would probably introduce a threshold of complexity for types of algebras after which we obtain a version of Conjecture 8.2.1.

We can give examples of types algebras that definitely lie within this threshold, even for more general quotients over ideals.

Proposition 8.2.5. *Let $\mathcal{C}$ be a monoidal category and let $\mathcal{I} \leq \mathcal{C}$ be a monoidal ideal. Any algebra $(\overline{A}, \overline{\mu})$ in $\mathcal{C}/\mathcal{I}$ can be obtained as the image of an algebra (A, μ) in $\mathcal{C}$, i.e. $\overline{A} = \mathrm{quot}_{\mathcal{I}}(A)$ and $\overline{\mu} = \mathrm{quot}_{\mathcal{I}}(A)$.*

Proof. This follows from the fact that $\mathrm{quot}_{\mathcal{I}}$ is a full functor that is surjective on objects. ■

Proposition 8.2.6. *Let $\mathcal{C}$ be a braided monoidal category that is enriched over a ring in which 2 is invertible, and let $\mathcal{I} \leq \mathcal{C}$ be an ideal. Any commutative or anti-commutative algebra in $\mathcal{C}/\mathcal{I}$ can be obtained as the projection of a commutative or anti-commutative algebra in $\mathcal{C}$ under $\mathrm{quot}_{\mathcal{I}}$.*

Proof. Let $(\overline{A}, \overline{\mu})$ be a commutative algebra in $\mathcal{C}/\mathcal{I}$. Proposition 8.2.5 implies that there exists an algebra (A, μ) in $\mathcal{C}$ such that $\mathrm{quot}_{\mathcal{I}}(A) = \overline{A}$ and $\mathrm{quot}_{\mathcal{I}}(\mu) = \overline{\mu}$. As 2 is invertible, we know that $\overline{\mu} \circ \overline{a}_2 = 0$ where $\overline{a}_2$ is the skew-symmetriser on $\overline{A} \otimes \overline{A}$. Note that $\overline{a}_2 = \mathrm{quot}_{\mathcal{I}}(a_2)$, where a_2 is the skew-symmetriser on $A \otimes A$. The equality $\overline{\mu} \circ \overline{a}_2 = 0$ implies that $\mu \circ a_2 \in \mathcal{I}$. It is then clear that $(A, \mu - \mu \circ a_2)$ is an algebra that is also mapped to $(\overline{A}, \overline{\mu})$ under $\mathrm{quot}_{\mathcal{I}}$, and that this algebra is commutative as $(\mu - \mu \circ a_2) \circ a_2 = 0$. ■

A Nederlandstalige Samenvatting

A.1 Inleiding

De theorie van (*symmetrische*) *tensorcategorieën*, zie bijvoorbeeld [Lan78; DM82; Del90; Del02; EGNO15], is een natuurlijke setting voor veel concepten in algebra. Het onderwerp van deze thesis is daar een voorbeeld van; we zijn geïnteresseerd in *niet-associatieve*, *niet-unitale* algebra's in symmetrische tensorcategorieën.

Tensorcategorieën zijn categorieën die bijna alle eigenschappen hebben die de categorie van vectorruimten over een veld heeft. Meer specifiek zijn de ruimten van morfismen in tensorcategorieën vectorruimten, bestaan er tensorproducten van objecten en morfismen $A \otimes B$ en $f \otimes g$, en zijn er dualen van objecten.

Recent toonde Arun S. Kannan aan in [Kan24] dat exceptionele Lie-superalgebra's kunnen geconstrueerd worden door gebruik te maken van Lie-algebra's in een "exotische" symmetrische tensorcategorie: de *Verlindecategorie* Ver_p. Deze categorie speelt ook een belangrijke rol binnen de classificatie van *pre-Tannakiaanse* symmetrische tensorcategorieën over algebraïsch gesloten velden van positieve karakteristiek. In karakteristiek nul toonde Pierre Deligne in [Del90; Del02] aan dat elke pre-Tannakiaanse symmetrische tensorcategorie over een algebraïsch gesloten veld van karakteristiek nul een representatiecategorie is van een affien supergroepschema. In [Ost20] toonde Victor Ostrik aan dat elke symmetrische fusiecategorie over een algebraïsch gesloten veld van karakteristiek $p > 0$ een representatiecategorie is van een affien groepschema in Ver_p. Dit toont aan dat Ver_p een gelijkaardige rol speelt in positieve karakteristiek als de categorie van supervectorruimten in karakteristiek nul.

Deze observatie van Kannan toont aan dat niet-associatieve algebra's in exotische tensorcategorieën zeer interessant kunnen zijn, en dit is dan ook de motivatie voor deze thesis. De constructie van Kannan, en tevens de constructie van Ver_p, verloopt via *semisimplificatie*. Semisimplificatie steunt op het feit dat veel tensorcategorieën (in het bijzonder symmetrische tensorcategorieën) zich gedragen als lokale ringen, in de zin dat zij een uniek maximaal tensorideaal hebben. Het quotiënt van de categorie hierover wordt dan de semisimplificatie van deze categorie genoemd, en dit levert een semisimplificatiefunctor op van de originele categorie naar de semisimplificatie. Deze semisimplificatiefunctor kunnen we toepassen op Lie-algebra's en dit levert dan Lie-algebra's op in de semisimplificatie. In het bijzonder werkt dit voor Ver_p, gedefinieerd als de semisimplificatie van de representatiecategorie van de affiene algebraïsche groep α_p, en omdat Ver_p de categorie van supervectorruimten als deelcategorie bevat kunnen we dan projecteren naar Lie-superalgebra's.

Naast constructies die gerelateerd zijn aan de constructie van Lie-algebra's van Kannan, blijven niet-associatieve algebra's in tensorcategorieën vrij mysterieus.

Om deze reden hebben wij in deze thesis vrij veel aandacht gespendeerd aan het semisimplificatieproces. Op sommige plekken zijn we in onze discussie dan ook dieper gegaan dan in de literatuur. Daarnaast hebben we ook veel tijd besteed aan het beschrijven van de nodige achtergrond om met niet-associatieve algebra's te kunnen werken.

A.2 Samenvatting

Hoofdstuk 1: Algemene Categorieën

We beginnen met het bespreken van *categorieën*, *functoren* en *natuurlijke transformaties*. Vervolgens bespreken we kort *limieten* en *colimieten*, die de begrippen product en coproduct veralgemenen. Daarna bespreken we de completering van categorieën met betrekking tot limieten of colimieten. Dit leidt ons op een natuurlijke manier naar de *Yoneda*- en *co-Yoneda*-lemma's. Functoren die limieten of colimieten behouden zijn van bijzonder belang, en *adjuncties* vormen belangrijke voorbeelden hiervan. Dit is dan ook het volgende dat we bespreken.

We sluiten het hoofdstuk af met een bespreking van *categorificatie*, een proces waarbij verzamelingtheoretische concepten worden vertaald naar hun categorische analogen.

Dit hoofdstuk bevat, net zoals de volgende drie hoofdstukken, geen nieuwe resultaten of bewijzen en sommige bewijzen zullen worden weggelaten.

Hoofdstuk 2: Abelse Categorieën

In dit hoofdstuk bespreken we categorieën uitgerust met een optelling op morfismen. Dergelijke categorieën worden *pre-additief* genoemd en als ze daarnaast ook directe sommen en een nulobject bezitten, heten ze *additief.* Vervolgens behandelen we additieve categorieën die *kernen* en *cokernen* toelaten, wat ons leidt tot de begrippen *Karoubiaanse* en *pre-abelse categorieën. Abelse categorieën* worden daarna geïntroduceerd als pre-abelse categorieën waarin de eerste isomorfie-stelling geldt. *Korte exacte rijen* spelen een fundamentele rol in de studie van abelse categorieën en laten toe structuurbehoudende functoren tussen dergelijke categorieën te definiëren.

We sluiten het hoofdstuk af met een bespreking van enkele van de belangrijkste stellingen in de theorie van abelse categorieën: de *Jordan-Hölder*-stelling, de *Krull-Schmidt*-stelling en het *lemma van Schur.*

Hoofdstuk 3: Monoïdale Categorieën

Het derde en belangrijkste hoofdstuk van het inleidende deel van deze thesis bespreekt *monoïdale categorieën.* We beginnen met het introduceren van de basistheorie van monoïdale categorieën. Dit zijn, ruwweg, categorieën uitgerust met een bifunctor $\otimes$, het zogenaamde *monoïdale product* of *tensorproduct*, die de objecten van de categorie met de structuur van een monoïde voorziet. In het bijzonder bestaat er een eenheid voor deze operatie, dat we het *monoïdale eenheidsobject* noemen.

Monoïdale categorieën zijn ontworpen om te lijken op de categorie van vectorruimten uitgerust met het gebruikelijke tensorproduct, en veel definities in de theorie zijn gemotiveerd door deze analogie. Zo zullen we de begrippen *dualen* van objecten en het *spoor* van een morfisme bespreken. Vervolgens bestuderen we *gevlochten* monoïdale categorieën. Een *vlechting* is een natuurlijk isomorfisme dat lijkt op de verwisselingsafbeelding $V \otimes W \to W \otimes V : v \otimes w \mapsto w \otimes v$ voor vectorruimten.

Hoofdstuk 4: Tensorcategorieën

In het laatste hoofdstuk van het inleidende deel bespreken we tensorcategorieën. Ruwweg zijn dit categorieën die op natuurlijke wijze een abelse structuur combineren met een monoïdale structuur die dualen bezit. Dit impliceert dat het monoïdale product en de dualisatiefunctoren structuurbehoudend moeten zijn als functoren tussen abelse categorieën.

We beginnen met het onderzoeken van de endomorfismen van het monoïdale eenheidsobject in een monoïdale categorie. Wanneer de categorie bovendien pre-additief is, vormen deze endomorfismen een ring, en elke andere morfismen-verzameling krijgt de structuur van een bimodule over deze ring.

Vervolgens richten we ons op categorieën die op natuurlijke wijze een abelse en een monoïdale structuur combineren, wat leidt tot de begrippen *multiringcategorieën* en *ringcategorieën.* Daarna voegen we dualen toe aan het geheel, wat resulteert in *multitensorcategorieën* en *tensorcategorieën.*

We sluiten af met een bespreking van symmetrische tensorcategorieën. We leggen in meer detail uit hoe de werking van de symmetrische groep op tensorproductmachten van objecten het mogelijk maakt om symmetrische en antisymmetrische producten te definiëren. We gaan ook kort in op de classificatie van pre-Tannakiaanse symmetrische tensorcategorieën, zoals eerder vermeld. Pre-Tannakiaanse symmetrische tensorcategorieën zijn zodanig dat tensormachten van objecten “subexponentieel groeien”.

Hoofdstuk 5: Semisimplificatie

In dit hoofdstuk bespreken we *semisimplificatie* in vrij veel detail. We beginnen met een herhaling van enkele basisbegrippen uit de theorie van *lokale ringen* en laten vervolgens zien hoe *onontbindbare* objecten in abelse categorieën kunnen worden gekarakteriseerd via lokale ringen: een object in een abelse categorie is onontbindbaar, wat wil zeggen dat het niet geschreven kan worden als een directe som van twee niet-nul objecten, als en slechts als zijn endomorfismenring een lokale ring is. Deze karakterisering speelt een belangrijke rol in de theorie van idealen in abelse categorieën.

Meer in het algemeen is elke pre-additieve categorie uitgerust met een optelling en een samenstelling, wat ervoor zorgt dat dit een categorificatie van een ring is. Dit stelt ons in staat om *idealen* in pre-additieve categorieën te definiëren als categorische analogieën van idealen in ringen. In het bijzonder bespreken we het radicaal, een speciaal ideaal dat lijkt op het Jacobson-radicaal van ringen. We laten zien dat het radicaal waardevolle informatie bevat over de vraag of een categorie semisimpel is, wat wil zeggen of elk object kan worden ontbonden als directe som van enkelvoudige (of simpele) objecten, objecten die onontbindbaar zijn en geen echte deelobjecten hebben.

Vervolgens behandelen we idealen in pre-additieve categorieën die ook een monoïdale structuur dragen, wat leidt tot het begrip *tensoridealen*. We tonen aan hoe dualen constructies van dergelijke idealen mogelijk maken en bewijzen het bestaan van een uniek maximaal tensorideaal, dat via het radicaal kan worden geconstrueerd. Dit resultaat illustreert een analogie tussen tensorcategorieën en lokale ringen. Het hoofdstuk eindigt met de beschrijving van de morfismen in dit maximale tensorideaal, zowel in het kader van de zogenaamde verwaarloosbare morfismen uit de literatuur als in een iets algemenere context.

Dit hoofdstuk bevat enkele originele bijdragen. In het bijzonder is Sectie 5.5.2 volledig origineel (zij het duidelijk geïnspireerd door bestaande literatuur). De resultaten in Secties 5.2-5.3 zijn grotendeels gebaseerd op het artikel [AKO02], terwijl Sectie 5.5.1 is gebaseerd op [**Etingof2018**].

Hoofdstuk 6: Algebra's in Monoïdale Categorieën

Zoals eerder vermeld, stellen monoïdale structuren op categorieën ons in staat om een aantal interessante algebraïsche objecten te construeren. Dit hoofdstuk is een voorbeeld van dat fenomeen: we bespreken *algebra's* in monoïdale categorieën, dit zijn objecten A uitgerust met een morfisme $\mu : A \otimes A \to A$, de *vermenigvuldiging*. Onze behandeling van algebra's is volledig algemeen; we veronderstellen niet dat algebra's *associatief* zijn of een *eenheid* hebben.

We bespreken ook modulen en idealen voor algebra's, en leggen uit hoe de werking van de symmetrische groep op tensormachten in symmetrische monoïdale categorieën leidt tot een natuurlijke veralgemening van Lie-algebra's.

We eindigen met een bespreking van *Hopf-algebra's* en hun *representatiecategorieën*, die aanleiding geven tot symmetrische tensorcategorieën. Dit is een zeer belangrijke constructie, zoals blijkt uit de classificatie van pre-Tannakiaanse symmetrische tensorcategorieën. Vervolgens bespreken we affiene groepschema's in symmetrische tensorcategorieën.

Dit hoofdstuk bevat één oorspronkelijke bijdrage die, voor zover wij weten, niet in de literatuur voorkomt: de constructie van een ideaal dat wordt gegenereerd door een subobject in algemene niet-associatieve algebra's, te vinden in Sectie 6.2.3.

Hoofdstuk 7: De Verlindecategorie

In dit zeer korte hoofdstuk bespreken we de *Verlindecategorie* Ver_p. We beginnen met enkele constructies voor deze categorie als semisimplificatie van een "klassieke" tensorcategorie. In het bijzonder bespreken we uitgebreid de constructie van Ver_p als de semisimplificatie van de representatiecategorie van de affiene algebraïsche groep α_p over een algebraïsch gesloten veld van karakteristiek $p > 0$. Vervolgens onderzoeken we de structuur van tensorproducten in deze categorie en wat dit ons vertelt over het tensorproduct in Ver_p. We sluiten af met een bespreking van subcategorieën van Ver_p.

Dit hoofdstuk bevat geen originele resultaten.

Hoofdstuk 8: Algebra's in de Verlindecategorie

Het laatste hoofdstuk betreft algebra's in de Verlindecategorie Ver_p. We onderzoeken de vlechtstructuur op de enkelvoudige objecten in deze categorie en leggen uit wat de implicaties daarvan zijn voor algebra's op deze objecten. We laten zien dat de enige enkelvoudige objecten die een algebra-structuur toelaten die met oneven dimensie zijn en dat de helft van deze algebra's aanleiding geeft tot wat wij *veralgemeende Lie-algebra's* noemen.

We voegen ook een korte bespreking toe over de constructie van Lie-algebra's via semisimplificatie en schetsen enkele vragen waarvan we hopen dat ze in toekomstig werk beantwoord zullen worden.

Alle resultaten in dit hoofdstuk zijn nieuw.

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